MATH 312 LECTURE 18 : Review for Midterm 2.

(Problems for the "extra problems" pdf).

P2 Least - squares

$$\left.\begin{aligned} &z = ax + by\sin(x) \\ &x = 0, -\pi/2, \pi/2, \pi, \\ &y = 3, -1, 1, -5, \\ &z = \tfrac{1}{2}, 1, 2, 3 \end{aligned}\right\}$$

1) Write equations, and in matrix form.

2) Find a, b ...

1)
$$\left.\begin{aligned} \tfrac{1}{2} &= 0 \\ 1 &= -\tfrac{\pi}{2}a - b\sin\left(\tfrac{-\pi}{2}\right) = \tfrac{-\pi}{2}a + b \\ 2 &= \tfrac{\pi}{2}a + b\sin\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}a + b \\ 3 &= \pi a - 5b \sin(\pi) = \pi a \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 0 \\ -\pi/2 & 1 \\ \pi/2 & 1 \\ \pi & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{b}}$$

2) To find a, b such that we obtain the least-squares solution, we solve the normal equations:

$$A^T A \vec{x} = A^T \vec{b}$$

Since A has orthogonal columns,

$$A^T A = \begin{bmatrix} \frac{3}{2}\pi^2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Also, $$A^T \vec{b} = \begin{bmatrix} 0 & -\pi/2 & +\pi/2 & \pi \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\pi + \pi - \frac{\pi}{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 7\pi/2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad a = x_1 = \frac{7\pi}{2} \cdot \frac{2}{3\pi^2} = \frac{7}{3\pi}$$

$$b = x_2 = 3/2$$

P7 $H \equiv z = 0$, $V \equiv z = y$ in $\mathbb{R}^3$

Basis for H: $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ ($\vec{e}_1$, $\vec{e}_2$).

1) Write $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ in the basis $\mathcal{E}$; i.e., find $\vec{u}|_{\mathcal{E}}$.

↳ $\vec{u} = 2\vec{e}_1 - \vec{e}_2 \Rightarrow \vec{u}|_{\mathcal{E}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2) Find an orthonormal basis $\mathcal{V}$ for V. ($V \leadsto -y + z = 0$)

↳ Basis for $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. (dim V = 2 !)

we were lucky; usually we need Gram-Schmidt.

so $\mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} / \sqrt{2} \right\} = \{ \vec{g}_1, \vec{g}_2 \}$

3) Write the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis $\mathcal{V}$.

↳ Since $\mathcal{V}$ is orthonormal,

$$\vec{v} = (\vec{v} \cdot \vec{g}_1)\vec{g}_1 + (\vec{v} \cdot \vec{g}_2)\vec{g}_2 \Rightarrow \vec{v}|_{\mathcal{V}} = \begin{bmatrix} \vec{v} \cdot \vec{g}_1 \\ \vec{v} \cdot \vec{g}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

4) The projection of vectors in V onto H is a linear transformation. Find its matrix using the basis V and $\mathcal{E}$.

↳ $$M_V^{\mathcal{E}} = \left[P_H \vec{g}_1 \big|_{\mathcal{E}} \quad P_H \vec{g}_2 \big|_{\mathcal{E}} \right].$$

↙ $\mathcal{E}$ is also orthonormal.

$$P_H \vec{g}_1 = (\vec{g}_1 \cdot \vec{e}_1)\vec{e}_1 + (\vec{g}_1 \cdot \vec{e}_2)\vec{e}_2 \Longrightarrow$$

$$\Rightarrow P_H \vec{g}_1 \big|_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P_H \vec{g}_2 \big|_{\mathcal{E}} = \begin{bmatrix} \vec{g}_2 \cdot \vec{e}_1 \\ \vec{g}_2 \cdot \vec{e}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow M_V^{\mathcal{E}} = \begin{bmatrix} 1 & 0 \\ 1 & 1/\sqrt{2} \end{bmatrix}.$$

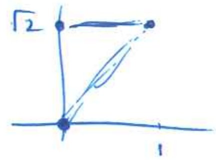
5) Find the area of the triangle with vertices $(0,0,0)$, $(1,1,1)$, $(0,1,1)$ ⇝ $\vec{p}_1, \vec{p}_2, \vec{p}_3$

↳ Notice that this triangle is on the plane V, for which we know an orthonormal basis. Indeed,

$$\vec{p}_1 \big|_V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{p}_2 \big|_V = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \quad \vec{p}_3 \big|_V = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix},$$

so the question becomes "find the area of the triangle with vertices $(0,0)$, $(1,\sqrt{2})$, $(0,\sqrt{2})$.

This is clearly $\frac{\sqrt{2}}{2}$ //



P11 P matrix that projects vectors of $\mathbb{R}^3$ to the plane $\underbrace{z=0}_{\equiv V}$.

1) Eigenvalues and eigenvectors of P?

↳ $\lambda_1 = 0$, $\lambda_2 = 1 = \lambda_3$.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underbrace{\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{C(P) = V}$$

$(N(P) = V^{\perp})$

2) Find the projection of $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

↳ $\vec{b} = 2\vec{v}_2 + 3\vec{v}_3 - \vec{v}_1 \Rightarrow P\vec{b} = 2P\vec{v}_2 + 3P\vec{v}_3 - \cancel{P\vec{v}_1} = 2\vec{v}_2 + 3\vec{v}_3 =$

$$= \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$ (easy to visualize in this case).

• Small questions (T/F)

1) If Q orthogonal matrix, then the corresponding transformation preserves lengths, i.e, length of $Q\vec{x}$ equal to length of $\vec{x}$.

↳ T: $\|Q\vec{x}\|^2 = (Q\vec{x})^T(Q\vec{x}) = \vec{x}^T \underbrace{Q^T Q}_{=I} \vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2$ //.

→ Similarly: A orthogonal, then $\lambda = 2$ cannot be eigenvalue.

↳ True: $A\vec{v} = 2\vec{v} \to \|A\vec{v}\| = 2\|\vec{v}\| \to\!\!\!\!/$.

$\|A\vec{v}\| = \|\vec{v}\|$

2) A square with orthogonal columns always has orthonormal rows.

↳ <u>True</u>: key word → square.

Orthonormal cols: $A^TA = I \to \det(A^TA) = 1 \to \{A \text{ square}\}$

$\to \det(A)^2 = 1 \to \det(A) = \pm 1 \neq 0 \to A$ invertible.

So,

$A^TAA^{-1} = A^{-1} \to A^T = A^{-1} \to \underbrace{AA^T = I}_{\Leftrightarrow \text{ orthonormal rows}}$ //.

3) There exists A diagonalisable with $\lambda = 0$ an eigenvalue.

↳ True; $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

↑ No relationship between being invertible and being diagonalisable !

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

non diag. invertible; non invertible, diag.

[4') A 3x3 matrix with eigenvalues 0, 1, -1 is always diagonalisable.]

4) A 3x3 matrix with eigenvalues 0, 0, 1 always has rank(A) = 1.

↳ False: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (typical case repeated eigenvalue with dim(eigenspace) = 1).

Remark: If rank(A) = 1, A 3x3, then A always has $\lambda_1 = 0 = \lambda_2$!

↳ of course: dim N(A) = 2.

related: 5) The only upper triangular 3x3 matrix with 1's on the diagonal which is diagonalisable is the identity.

P6 (From homework 5).

$x_1 + x_2 + x_3 - x_4 = 0. \equiv S$

1) Find orthonormal basis for S and $S^\perp$. dim $S = 3$!

→ For S: apply Gram-Schmidt ⇝ $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

typical mistakes.
- → Check that $\vec{q}_1 \cdot \vec{q}_2 = 0$, $\vec{q}_1 \cdot \vec{q}_3 = 0$, $\vec{q}_2 \cdot \vec{q}_3 = 0$!
- → Normalize: $\|\vec{q}\| = 1$.

→ For $S^\perp$: very easy!

dim $S^\perp = 1$, and $\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \in S^\perp$ since $\begin{bmatrix}1 & 1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\-x_4\end{bmatrix} = 0$ for all $\vec{x} \in S$.

So $\left\{\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}/2\right\}$ is orthonormal basis for $S^\perp$

$\vec{u}_1 \equiv$ ↑ most forgot to normalize.

3) Find $\vec{b}_1 \in S$, $\vec{b}_2 \in S^\perp$ / $\vec{b}_1 + \vec{b}_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \equiv \vec{b}$.

Easiest way: $\vec{b}_2 = P_{S^\perp}\vec{b} = (\vec{b}\cdot\vec{u})\vec{u} = \frac{2}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\frac{1}{2} = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\frac{1}{2}$

Then, $\vec{b}_1 = \vec{b} - \vec{b}_2 = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}$ (check that $\vec{b}_1 \in S$!).

Remark: We didn't use the basis for S.
(the hard one to compute).

4) Point in S closest to $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

↳ That is, $P_S \vec{v}$.

Two recommended ways:

→ $P_S \vec{v} = (\vec{v} \cdot \vec{q}_1)\vec{q}_1 + (\vec{v} \cdot \vec{q}_2)\vec{q}_2 + (\vec{v} \cdot \vec{q}_3)\vec{q}_3$ (using an orthonormal basis.).

or

→ $P_S \vec{v} = \vec{v} - P_{S^\perp} \vec{v} = \vec{v} - (\vec{u}_1 \cdot \vec{v})\vec{u}_1$ (easiest and fastest).

~~Third way: Find the~~

"Third way" → use the "traditional" projection matrix starting with the ~~basis~~ original basis for S... crazy!

↳ We study orthonormal basis for a reason.