MT312: LECTURE 18.

ODEs and difference equations.

Example: Solve $x_1'(t) = 7x_1(t) + 5x_2(t)$, $x_2'(t) = -10x_1(t) - 8x_2(t)$

with $x_1(0) = 1$, $x_2(0) = 2$.

Solution: $\vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}}_{A} \vec{x}(t)$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\vec{x}(t) = e^{At}\vec{x}(0) \rightarrow$ Find e^{At}.

We first try to diagonalise A:

$$A - \lambda I = \begin{bmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{bmatrix} \rightarrow -(7-\lambda)(8+\lambda) + 50 = \lambda^2 + \lambda - 6 = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda = \frac{-1 \pm \sqrt{25}}{2} \rightarrow \lambda_1 = -3,\ \lambda_2 = 2 \quad (\Rightarrow \text{we can diagonalise})$$

Eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, $S = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

and so

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix}-1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}e^{-3t} & 0\\ 0 & e^{2t}\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$$

<u>Remark</u>: The solution can be written as follows

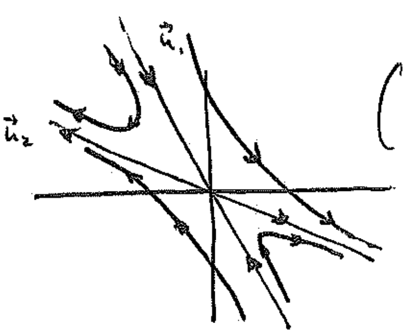
$$\vec{x}(t) = e^{At}\vec{x}_0 = \underset{S}{\begin{bmatrix}-1 & 1\\ 2 & -1\end{bmatrix}}\underset{D}{\begin{bmatrix}e^{-3t} & 0\\ 0 & e^{2t}\end{bmatrix}}\underbrace{\underset{S^{-1}}{\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}}\underset{\vec{x}_0|_E}{\begin{bmatrix}1\\ 2\end{bmatrix}}}_{\vec{x}_0|_S} =$$

$$= \begin{bmatrix}-e^{-3t} & e^{2t}\\ 2e^{-3t} & -e^{2t}\end{bmatrix}\underbrace{\begin{bmatrix}3\\ 4\end{bmatrix}}_{\vec{x}_0|_S} = 3e^{-3t}\underbrace{\begin{bmatrix}-1\\ 2\end{bmatrix}}_{\vec{u}_1} + 4e^{2t}\underbrace{\begin{bmatrix}1\\ -1\end{bmatrix}}_{\vec{u}_2}$$

coordinates of $\vec{x}_0$ in $\{\vec{u}_1, \vec{u}_2\}$

This shows us how the solution evolves in time: the $\vec{u}_1$ component goes to zero since its corresponding eigenvector is negative.

Phase portrait:



(Saddle point)

↳ Always the case when $\lambda_1 \cdot \lambda_2 < 0$ (i.e., distinct signs)

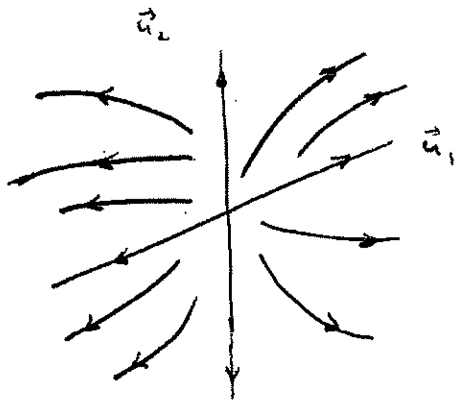
<u>Remark</u>: If A is diagonalizable, the solution can always be written as

$$\vec{x}(t) = S e^{\Lambda t} S^{-1} \vec{x}_0 = \left[\vec{u}_1 \,\middle|\, \vec{u}_2 \right] \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \vec{x}_0|_S =$$

$$= c_1 e^{\lambda_1 t} \vec{u}_1 + c_2 e^{\lambda_2 t} \vec{u}_2 \,, \quad \text{where } \vec{x}_0|_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

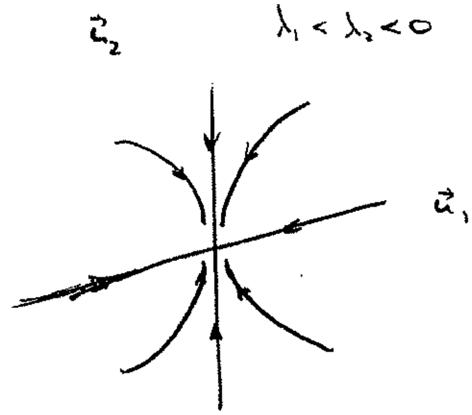
Depending on the <u>signs</u> of λ_1, λ_2, different phase portraits arise

$\lambda_1 > \lambda_2 > 0$



(everything grows, but faster in $\vec{u}_1$ direction)

$\lambda_1 < \lambda_2 < 0$



(everything goes to zero, but the $\vec{u}_1$ component dissapears faster)

⋮

• Exercise: 2^{nd} order linear differential equation.

Solve $x''(t) = -x(t) + 2x'(t)$ with $x(0) = 2$, $x'(0) = 1$ $\Bigg\}$.

Sol.: Right now, we only know how to solve first order linear systems.

Let's convert this 2^{nd} order diff. eq. into a 1^{st} order system:

Denote $\left.\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x'(t) \end{aligned}\right\}$

Then, $x_1'(t) = x'(t) = x_2(t)$

$$x_2'(t) = \underbrace{x''(t) = -x(t)}_{\text{use the equation}} + \underbrace{2x'(t) = -x_1(t) + 2x_2(t)}_{\text{introduce the new variables}}$$

So, in matrix form,

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}'}_{\vec{x}'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)}$$

and thus $\vec{x}(t) = e^{At}\vec{x}_0$.

Remark: This idea can be applied to any n-order lin. diff. equation with constant coeff. to produce ~~a~~ ~~a~~ a 1^{st} order lin. system (of n equations)

$$a_0 x(t) + a_1 x'(t) + \dots + a_n x^{n)}(t) = 0$$

$\hookrightarrow$ Define

$$\left.\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x'(t) \\ &\vdots \\ x_n(t) &= x^{n-1)}(t) \end{aligned}\right\} \quad \vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & \ddots & \vdots \\ \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \dots & & \frac{-a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Let's finish the exercise. We need e^{At}.

Try to diagonalise:

Eigenvalues of A: $\det(A - \lambda I) = 0 \Leftrightarrow -\lambda(2-\lambda) + 1 = 0 \Leftrightarrow$

$$\Leftrightarrow \lambda^2 - 2\lambda + 1 = 0 \Leftrightarrow (\lambda - 1)^2 = 0$$

$$\lambda = 1 \text{ (repeated)}$$

Eigenvectors: $(A - I)\vec{u} = \vec{0} \rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ has rank 1! Only one eigenvector.

So A cannot be diagonalised.

We have to use definition of $e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$

However,

$$A^2 = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}, \dots \quad \text{infinite terms?!}$$

- <u>Nice fact</u>: (Cayley-Hamilton Theorem)

$$(A-I)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[A matrix A always satisfies its own characteristic equation: $\det(A-\lambda I) = 0$]

(here $\det(A-\lambda I) = (\lambda-1)^2 = 0$)

So we can write $A = I + (A-I)$ and so,

$$e^{At} = e^{It + (A-I)t} \overset{(*)}{=} e^{It} e^{(A-I)t} =$$

$$= e^t I \left(I + (A-I)t + \underbrace{\cancel{(A-I)^2}}_{0} \frac{t^2}{2!} + \cancel{\dots} \right) =$$

$$= e^t \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \right) = e^t \begin{bmatrix} 1-t & t \\ -t & 1-t \end{bmatrix}$$

((*) $e^{A+B} = e^A e^B \iff A$ and B commute (i.e. $AB = BA$))

We conclude that

$$\vec{x}(t) = e^{At}\vec{x}_0 = e^t\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = e^t\begin{bmatrix} 2-2t+t \\ -2t+1+t \end{bmatrix} = e^t\begin{bmatrix} 2-t \\ 1-t \end{bmatrix} =$$

$$= e^t\underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{x}_0} - te^t\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{u}_1}$$

(of course $\vec{x}(0) = e^0\vec{x}_0 - \vec{0} = \vec{x}_0$)

V) Difference equations (sequences)

$$\left.\begin{aligned} \vec{x}(k) &= A\vec{x}(k-1) \\ \vec{x}(0) &= \vec{x}_0 \end{aligned}\right\}$$

(... like differential equations with discrete time... bla blah..)

Solution: $\vec{x}(1) = A\vec{x}(0) = A\vec{x}_0$

$$\vec{x}(2) = A\vec{x}(1) = AA\vec{x}_0 = A^2\vec{x}_0$$

$$\vdots$$

$$\boxed{\vec{x}(k) = A^k\vec{x}_0}$$

If $A = SDS^{-1}$, then

$$\vec{x}(k) = SD^kS^{-1}\vec{x}_0 = S\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}S^{-1}\vec{x}_0$$

<u>Example</u>: Virahanka (or Fibonacci) numbers.

Sequence $0, 1, 1, 2, 3, 5, 8, \ldots, N_k, N_{k+1}, N_{k+2}, \ldots$

N_k ? $\dfrac{N_k}{N_{k-1}}$?

<u>Sol</u>: The sequence is defined by $N_{k+2} = N_{k+1} + N_k$.

This is a 2^{nd} order difference equation (as it involves two steps).

But we "already know" how to solve that problem: convert it into two 1^{st} order equations.

Denote

$$\vec{x}(k) = \begin{bmatrix} N_{k+1} \\ N_{k+2} \end{bmatrix}, \text{ then}$$

$$\underbrace{\vec{x}(k) = \begin{bmatrix} N_{k+1} \\ N_{k+2} \end{bmatrix}}_{\vec{x}(k)} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} N_k \\ N_{k+1} \end{bmatrix}}_{\vec{x}(k-1)} \quad \text{with } \vec{x}(0) = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the solution is

$\vec{x}(k) = A^k \vec{x}_0 \longrightarrow A^k$? Diagonalize:

1) Eigenvalues: $\det(A-\lambda I)=0 \Leftrightarrow$

$$\Leftrightarrow \lambda^2-\lambda-1=0 \Leftrightarrow \lambda = \frac{1\pm\sqrt{5}}{2} \quad \begin{array}{l} \nearrow \lambda_1 = \frac{1+\sqrt{5}}{2} \\ \searrow \lambda_2 = \frac{1-\sqrt{5}}{2} \end{array}$$

2) Eigenvectors: $(A-\lambda_1 I) = \begin{bmatrix} -\lambda_1 & 1 \\ 1 & 1-\lambda_1 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ (note that $1+\lambda_1-\lambda_1^2=0$)

Analogously, $\vec{u}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

In conclusion,

$$\vec{x}(k) = \underbrace{\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}}_{S} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} (S^{-1}\vec{x}_0) \quad \text{where } S^{-1}\vec{x}_0 = \vec{x}_0|_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

(just a matter of computing)

thus

$$\vec{x}(k) = \left[\lambda_1^k \vec{u}_1 \,\middle|\, \lambda_2^k \vec{u}_2 \right] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \lambda_1^k \vec{u}_1 + c_2 \lambda_2^k \vec{u}_2 .$$

Therefore,

$$N_{k+2} = c_1 \lambda_1^{k+1} + c_2 \lambda_2^{k+1} \quad (2^{\text{nd}} \text{ component of } \vec{x}(k))$$

and

$$\frac{N_{k+2}}{N_{k+1}} = \frac{c_1\lambda_1^{k+1} + c_2\lambda_2^{k+1}}{c_1\lambda_1^k + c_2\lambda_2^k} = \frac{c_1\lambda_1 + c_2\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^k}{c_1 + c_2(\lambda_2/\lambda_1)^k} \Rightarrow \lim_{k\to\infty} \frac{N_{k+2}}{N_{k+1}} = \lambda_1 = \frac{1+\sqrt{5}}{2}$$

("golden ratio")

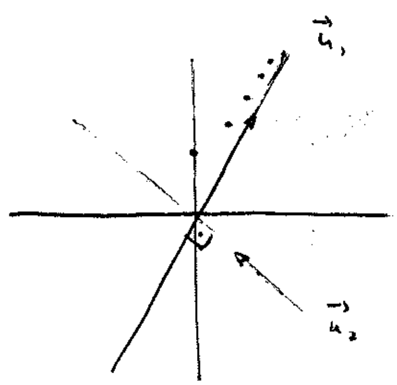
(because $\lambda_2 < \lambda_1$)

Forgetting about Virahanka or Fibonacci, we could think of

$$\vec{x}(k) = \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(k-1) \\ y(k-1) \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Bigg\} \text{ as a system,}$$

whose solution $\vec{x}(k) = c_1 \lambda_1^k \vec{u}_1 + c_2 \lambda_2^k \vec{u}_2$ can be drawn as "time" evolves (k increases):

$$\vec{x}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{x}(2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \ldots$$



The $\vec{u}_2$ component goes to zero as $|\lambda_2| < 1$.

→ Notice that $\vec{u}_1 \cdot \vec{u}_2 = 1 + \lambda_1 \lambda_2 = \frac{1+\sqrt{5}}{2} \frac{1-\sqrt{5}}{2} = 1 - 1 = 0$, that is, $\vec{u}_1 \perp \vec{u}_2$.

This is not a coincidence: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is symmetric with real entries.