Important and useful:

det(A) = product of eigenvalues.

trace(A) = sum of eigenvalues (trace(A) = sum of diagonal entries).

$\boxed{\begin{array}{l}\text{MATH 312} \\ \text{LECTURE 17}\end{array}}$: Diagonalization and ODEs.

III) Diagonalization.

• **Theorem**: If A has n l.i. eigenvectors $\vec{u}_1, \dots, \vec{u}_n$, then

$$A = SDS^{-1} \quad \text{where} \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ (eigenvalues)}$$

$$S = \left[\vec{u}_1 \,\middle|\, \cdots \,\middle|\, \vec{u}_n \right] \text{ (eigenvectors)}$$

Proof: $\vec{u}_1, \dots, \vec{u}_n$ l.i. $\Rightarrow \exists S^{-1}$.

So just need to check $AS = SD$:

$$AS = A\underbrace{\left[\vec{u}_1 \,\middle|\, \cdots \,\middle|\, \vec{u}_n \right]}_{S} = \left[\lambda_1 \vec{u}_1 \,\middle|\, \cdots \,\middle|\, \lambda_n \vec{u}_n \right] = \underbrace{\left[\vec{u}_1 \,\middle|\, \cdots \,\middle|\, \vec{u}_n \right]}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \quad //.$$

Examples: Find diagonalization of A :

1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Eigenvalue : $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$ $\lambda_1 = 1$, $\lambda_2 = 2$.

Remark: For triangular matrices, the eigenvalues are the diagonal elements.

(of course! recall det of triangular matrix)

Eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,
$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(same order!)

2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenvalues $\rightarrow$ $\lambda_1 = 1 = \lambda_2$ (repeated)

Eigenvectors: $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

rank$(A-I) = 1$, so its nullspace has dimension 1. $\rightarrow$

$\rightarrow$ Only 1 eigenvector! It is not possible to diagonalise A.

$\rightarrow$ • Remark: Possible cases (Summary)

1) All eigenvalues are distinct: Then all eigenvectors are l.i., therefore we can diagonalize. [(*) proof in -143-] (optional)

2) Some repeated eigenvalues: We have to compute the nullspace of those repeated eigenvalues.

↳ If we obtain full set of eigenvectors $\rightarrow$ diagonalise ok ✓.

↳ If we don't $\rightarrow$ diagonalisation not possible ✗.

(*) (proof) Distinct eigenvalues $\Rightarrow$ l.i. eigenvectors : (not done in class).

Assume it is false, i.e.,

$\vec{u}_1, \vec{u}_2$ are two eigenvectors corresponding to $\lambda_1 \neq \lambda_2$ and they are l.d. Then,

[1] $c_1 \vec{u}_1 + c_2 \vec{u}_2 = \vec{0}$ for some c_1, c_2 <u>not</u> both zero.

But, multiply [1] by A:

$c_1 A \vec{u}_1 + c_2 A \vec{u}_2 = 0 \Rightarrow c_1 \lambda_1 \vec{u}_1 + c_2 \lambda_2 \vec{u}_2 = \vec{0}$

Multiply [1] by λ_2:

$c_1 \lambda_2 \vec{u}_1 + c_2 \lambda_2 \vec{u}_2 = \vec{0} \longrightarrow c_1(\lambda_1 - \lambda_2)\vec{u}_1 = 0 \Rightarrow \underline{c_1 = 0}$

$\lambda_1 \neq \lambda_2$

$\vec{u}_1 \neq \vec{0}$ (by def. of eigenvector)

Multiply [1] by λ_1:

$c_1 \lambda_1 \vec{u}_1 + c_2 \lambda_1 \vec{u}_2 = \vec{0} \longrightarrow c_2(\lambda_2 - \lambda_1)\vec{u}_2 = \vec{0} \Rightarrow \underline{c_2 = 0}$

So we conclude $c_1 = c_2 = 0$! Contradiction.

(the proof can be extended to more eigenvalues using the same idea...)

• <u>Exercise:</u> $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ find diagonalisation.

Hint: $\operatorname{rank}(A) = 1$.

<u>Sol</u>:

$\operatorname{rank}(A) = 1 \Rightarrow \dim N(A) = 2 \Rightarrow \lambda_1 = \lambda_2 = 0$

Now,

$\operatorname{trace}(A) = 6 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3 = 6$.

Eigenvectors:

$$A - \lambda_1 I = A \sim \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} \sim \quad \vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} -4 & 1 & 2 \\ 4 & -4 & 4 \\ 2 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 3/2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

free

$$\rightarrow \vec{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

so

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^{-1} //.$$

IV) Linear differential equations with constant coeff.

Consider the 1-d differential equation

$$\left.\begin{aligned} x'(t) &= \alpha x(t) \\ x(0) &= x_0 \end{aligned}\right\} \qquad \left(x'(t) \equiv \frac{dx}{dt}(t) \right)$$

$x(0) = x_0 \rightarrow$ (Initial condition)

To solve it, we simply integrate it:

$$\int_0^t \frac{x'(\tau)}{x(\tau)} d\tau = \int_0^t \alpha d\tau \Rightarrow \ln x(\tau) \Big]_0^t = \alpha t \Rightarrow$$

$$\Rightarrow \ln(x(t)) - \ln(x(0)) = \ln\left(\frac{x(t)}{x_0}\right) = \alpha t \Rightarrow \boxed{x(t) = x_0 e^{\alpha t}}$$

The sign of the constant α tell us if the solution grows exponentially to infinity or decreases exponentially to zero.

- Remark: We can always check the solution of a diff. eq.:

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(x_0 e^{\alpha t}\right) = x_0 \alpha e^{\alpha t} = \alpha x(t) \quad \text{(satisfies the eq.)}$$

· The previous equation provides a simple (but silly) model for growth population:

$$p'(t) = \alpha\, p(t)$$

← current population

rate of growth at time t · fraction of reproducing people

"The more we are, the more kids we have". $(\alpha \in [0,1])$

But this always gives $p(t) = p_0 e^{\alpha t}$! (Exponential growth).

· We can improve the model taking into account the resources:

$r(t) \equiv$ resources at time t
$p(t) \equiv$ population at time t

and let's assume that

$p'(t)$ is proportional to $r(t)$ and $p(t)$ (increasing)
$r'(t)$ " " to $p(t)$ (decreasing)

then,

$$\left.\begin{aligned} p'(t) &= \alpha p(t) + \beta r(t) \\ r'(t) &= -\gamma p(t) \end{aligned}\right\} \quad (\text{with } \alpha, \beta, \gamma > 0 \text{ constants}).$$

We can write this in matrix form:

$$\begin{bmatrix} p(t) \\ r(t) \end{bmatrix}' = \begin{bmatrix} \alpha & \beta \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ r(t) \end{bmatrix}$$

- <u>Def</u>: We call linear system of differential equations with constant coefficients those of the form

  $\vec{x}'(t) = A\vec{x}(t)$ with A $n \times n$ matrix and

  $\vec{x}(0) = \vec{x}_0$ initial condition.

How to solve it?

First, note that if $A = D$ diagonal, then the system $\vec{x}'(t) = D\vec{x}(t)$ is in reality n separated equations:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \sim \begin{array}{l} x_1'(t) = \lambda_1 x_1(t) \Rightarrow x_1(t) = x_1(0)e^{\lambda_1 t} \\ x_2'(t) = \lambda_2 x_2(t) \Rightarrow x_2(t) = x_2(0)e^{\lambda_2 t} \end{array} \Bigg\|$$

or in matrix notation,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

This motivates the following definition:

• Def: Exponential matrix e^A

$$e^A \underset{\text{def.}}{=} I + A + \frac{A^2}{2!} + \dots = \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

(recall $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ when $x \in \mathbb{R}$)

Remarks:

1) If $A = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $e^D = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$

$$\hookrightarrow e^D = \sum_{j=0}^{\infty} \frac{1}{j!} D^j = \sum_{j=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^j}{j!} & 0 \\ 0 & \frac{\lambda_2^j}{j!} \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\infty} \frac{\lambda_1^j}{j!} & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{\lambda_2^j}{j!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$$

2) If A is diagonalizable, i.e., $A = SDS^{-1}$, then

$$\left\| e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!} = \sum_{j=0}^{\infty} \frac{1}{j!} SD^jS^{-1} = S\left(\sum_{j=0}^{\infty} \frac{D^j}{j!}\right) S^{-1} = Se^DS^{-1} \right\|$$

3) Example of nondiag: (not done yet)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to e^{At} \underset{*}{=} e^{\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}} e^{\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}}.$$

3) $\frac{d}{dt}(e^{At}) = Ae^{At}$

$$\hookrightarrow \frac{d}{dt}\left(\sum_{j=0}^{\infty} \frac{A^j}{j!} t^j\right) = \frac{d}{dt}\left(I + \sum_{j=1}^{\infty} \frac{A^j t^j}{j!}\right) = \sum_{j=1}^{\infty} \frac{A^j j}{j!} t^{j-1} =$$

$$= A \sum_{j=1}^{\infty} \frac{A^{j-1}}{(j-1)!} t^{j-1} = A \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k = Ae^{tA} //.$$

<u>Conclusion</u>: The system $\left.\begin{array}{l} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(0) = \vec{x}_0 \end{array}\right\}$ is solved by

$$\vec{x}(t) = e^{At}\vec{x}_0$$

<u>Proof</u>: Let's check that $\vec{x}(t) = e^{At}\vec{x}_0$ verifies the equation:

$$\vec{x}'(t) = (e^{At}\vec{x}_0)' = Ae^{At}\vec{x}_0 = A\vec{x}(t) //$$

(and obviously verifies the initial condition $\vec{x}(0) = e^{\overset{\text{zero matrix}}{0}}\vec{x}_0 =$
$= I\vec{x}_0 = \vec{x}_0$)

• <u>Remark</u>: The solution $\vec{x}(t) = e^{At}\vec{x}_0$ is valid for any A //.