(Start of lecture 16)

Determinant as volume

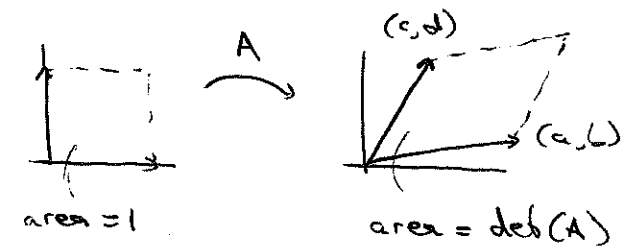
One can see that the volume (n-dimensional) satisfies the same properties of the determinant, so, since that function was unique, they are the same thing.

That is, if

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ represent a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (with canonical basis), then

$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

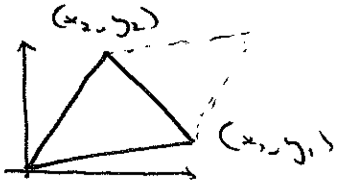
$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}c\\d\end{bmatrix}$



(Note: see google for some nice visualizations in 3D).

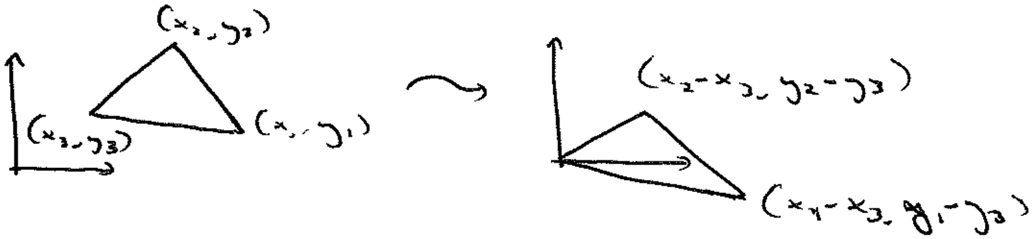
<u>Remark</u>: Notice that if we scale a column by λ, the det. gets multiplied by λ. This corresponds to multiplying one ~~the~~ side of the square by λ, the volume (area in 2D) gets multiplied by λ.

We can use this to find the formula for the area of a general triangle knowing the vertices:



$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

Similarly, for

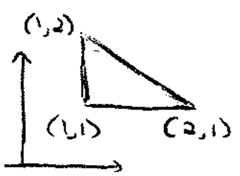


$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} =$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_3 & x_3 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} =$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_3 & y_3 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_3 & x_3 \\ y_1 & y_2 \end{vmatrix} + \frac{1}{2}\cancel{\begin{vmatrix} x_3 & x_3 \\ y_3 & y_3 \end{vmatrix}}$$

(This is just to check that it coincides ~~with~~ with the formula on the book. For a real example $x_1, \dots, y_3$ are all numbers! So we just compute the 2x2 determinant (instead of the 3x3 in the book):



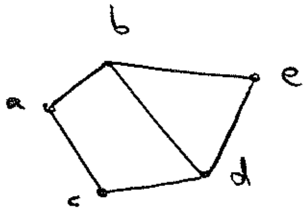
Area $= \frac{1}{2}$. Using the formula: $\frac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$ ✓)

MATH 312
LECTURE 16

: Eigenvalues and eigenvectors.

I) Introduction.

Let's try to solve the following problem. Given the graph



how many paths connecting a to e?

We can define the graph using the adjacency matrix:

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc} a & b & c & d & f \end{array}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}$$

There is a 1 only if you can get from one node to the other with a path of length one.

Look at A^2:

$$A^2 = \begin{bmatrix} 2 & 0 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Each entry "coincides" with the number of paths of length 2 connecting the two nodes.

One can check that the entry i, j of A^k is the n° of paths of length k joining nodes i and j.

So, if we know A^k for all k we know how to compute the number of paths... But how to compute A^{1000}, for example, if the graph represents something bigger, like cities in US?

→ The main purpose of eigenvalues/eigenvectors (that arises in most applications) is to compute powers of A in an efficient way.

• Let's start with diagonal matrices:

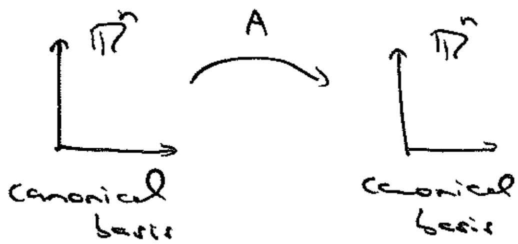
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \to A^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \to \dots \to$$

$$\to A^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \quad \underline{\underline{\text{easy}}}. \quad (\text{same for } A \ n \times n \text{ matrix})$$

But, recall that an $n \times n$ matrix represent a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ (with chosen basis, for example, the canonical one).

Question: Can we find a new basis for $\mathbb{R}^n$ in which a given general matrix A becomes diagonal?

That is, if

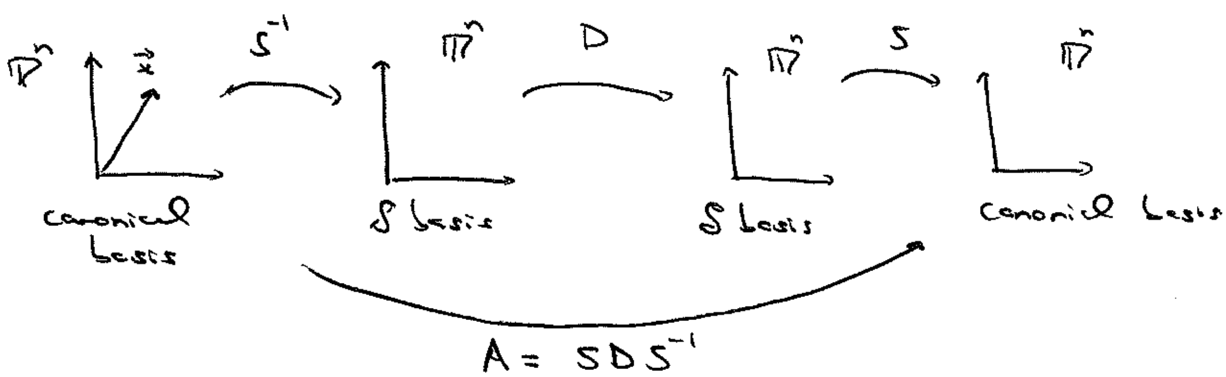


A non diagonal,

can we find a basis $\mathcal{S} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ such that the linear application above is represented by a diagonal basis?

- If the answer is yes, we would have an invertible matrix

$$S = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n \right] \quad \text{such that} \quad A = SDS^{-1}$$



Note that $S = [\vec{u}_1 | \dots | \vec{u}_n]$ is the matrix that changes coordinates of a vector written in the S basis to the canonical one:

$$\vec{x} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n = \underbrace{\begin{bmatrix} \vec{u}_1 & \Big| & \dots & \Big| & \vec{u}_n \end{bmatrix}}_{S} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{x}\Big|_{\substack{\text{canonical} \\ \text{basis}}}$$

vectors written in the canonical basis.

$\vec{x}|_S \equiv$ coordinates of $\vec{x}$ in the basis S

- Going back, we wanted to compute A^k. If we assume that we can obtain $A = SDS^{-1}$, then

$$A^k = SD\underbrace{S^{-1} S}D\underbrace{S^{-1} S}DS^{-1} \dots SDS^{-1} = SDDD\dots DS^{-1} = SD^kS^{-1} \quad //.$$

diagonal, so easy!

- OK, so how do we find such matrices S, D?

Let's see what characterizes diagonal matrices.

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ Notice what D does to the basis vectors:

$$\left.\begin{array}{l} D\vec{e}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} = \lambda_1 \vec{e}_1 \\ D\vec{e}_2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix} = \lambda_2 \vec{e}_2 \end{array}\right\}$$

That is, we want to find those vectors $\vec{u}$ that don't change their direction when multiplied by A:

$$\boxed{A\vec{u} = \lambda\vec{u}}$$

Those $\vec{u}$ will give the basis on which the linear application given by A (in the canonical basis) becomes diagonal.

II) Eigenvalues and eigenvectors

- Def: We say that $\vec{u} \neq \vec{0}$ is an eigenvector of A with corresponding eigenvalue λ if
  $A\vec{u} = \lambda\vec{u}$.

- How to find λ and $\vec{u}$?

$A\vec{u} = \lambda\vec{u}$ is nonlinear, since both λ and $\vec{u}$ are unknowns.

But,

$A\vec{u} = \lambda\vec{u} \iff A\vec{u} - \lambda\vec{u} = \vec{0} \iff (A - \lambda I)\vec{u} = \vec{0}$. That is,

$\vec{u} \in N(A - \lambda I)$ and, $\vec{u} \neq \vec{0}$ (by definition), so

$\|A - \lambda I$ has to be singular$\|$ (its nullspace is not the trivial $\{\vec{0}\}$).

Therefore,

$$\|\underbrace{\det(A - \lambda I)}_{\substack{\text{Characteristic} \\ \text{polynomial of } A \\ \text{(degree } n)}} = 0\|$$ → This gives us an equation on λ (of order n)

↓ Solve it to find

n eigenvalues (→ Can be complex! → Can be repeated!)

<u>Example:</u>

1) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \longrightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

So $\det(A - \lambda I) = (1-\lambda)(4-\lambda) - 4 = 4 + \lambda^2 - 5\lambda - 4 = \lambda(\lambda - 5)$,

$\det(A - \lambda I) = 0 \iff \lambda(\lambda - 5) = 0 \iff \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 5 \end{cases}$.

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \to A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \Rightarrow$

$$\det(A-\lambda I) = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1 \Leftrightarrow \begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$$

<u>Remark</u>: It is not a surprise that this matrix doesn't have real eigenvalues. Why?

$$A\vec{u} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \rightsquigarrow \text{it is a rotation of } \vec{u} \text{ by } 90^\circ$$

That is, any vector $\vec{u}$ gets rotated by A!

But an eigenvector can only be scaled: $A\vec{u} = \lambda\vec{u}$!!

- So, we first find the λ's by solving an equation of degree n. Then we find the eigenvectors:

$(A - \lambda_1 I)\vec{u}_1 = \vec{0} \rightsquigarrow \vec{u}_1 \in N(A - \lambda_1 I)$ we already know how to find the $\vec{u}$'s!

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\lambda_1 = 0$, $\lambda_2 = 5$. Eigenvectors?

$$A - \lambda_1 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \to (A - \lambda_1 I)\vec{u} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(any multiple is valid)

$$A - \lambda_2 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \to \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Exercise: If A has eigenvalues λ_1, λ_2, what are the eigenvalues of A^2? and of $A + I$? of A^{-1}?

Sol: Let $\vec{u}_1, \vec{u}_2$ be the corresponding eigenvectors of A:

$$A\vec{u}_1 = \lambda_1 \vec{u}_1$$

Then,
- $AA\vec{u}_1 = \lambda_1 A\vec{u}_1 = \lambda_1^2 \vec{u}_1 \Rightarrow \lambda_1^2$ is an eigenvalue of A^2 (with same eigenvector $\vec{u}_1$)

- $(A + I)\vec{u}_1 = A\vec{u}_1 + \vec{u}_1 = \lambda_1 \vec{u}_1 + \vec{u}_1 = \underbrace{(\lambda_1 + 1)}_{\text{eigenvalue of } A+I} \vec{u}_1$.

- $A^{-1}A\vec{u}_1 = \lambda_1 A^{-1}\vec{u}_1 \Rightarrow A^{-1}\vec{u}_1 = \frac{1}{\lambda_1}\vec{u}_1$

  ↳ eigenvalue of A^{-1}.

• Remark: If A has an eigenvalue equal to zero, the A does not have an inverse.

$\left(\hookrightarrow A\vec{u} = \vec{0} \text{ for } \vec{u} \neq \vec{0} \right)$

the corresponding eigenvector is in $N(A)$.

• Remark: Eigenvectors corresponding to nonzero eigenvalues are trivially in $C(A)$.

$A\vec{u} = \underbrace{\lambda\vec{u}}$ this is "output" by A (it's a linear comb. of its columns).

• Exercise: If A has an eigenvalue λ and B an eig. β, is $\lambda\beta$ an eigenvalue of AB?

False in general.

$$\left(\left.\begin{array}{l} A\vec{u}_1 = \alpha\vec{u}_1 \\ B\vec{u}_2 = \beta\vec{u}_2 \end{array}\right\} \to AB\vec{u}_2 = A\beta\vec{u}_2 \,! \right)$$

this is not, in general, an eigenvector of A.