MATH 312
LECTURE 15

: Determinants

This lecture is intended as a reminder of known things.

[*] We started the lecture solving two problems related to orthonormal basis / G-S

I] Properties

• Def: Determinant

The determinant is the only function from the vector space of $\underline{\underline{n \times n}}$ matrices to real numbers,

$$\left.\begin{array}{rcl} \det : n\times n \text{ matrices} & \longrightarrow & \mathbb{R} \\ A & \longmapsto & \det(A) \equiv |A| \end{array}\right\}$$

that satisfies the following three properties:

1) Normalization: $\det(I) = 1$ (I identity matrix of size $n \times n$)

2) Antisymmetry: Exchange of two rows changes the sign.

Example: $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3) Multilinearity: The determinant is a linear function of each row separately (i.e., keeping the rest constant).

That is, multilinearity means that:

3.1) Scaling a row scales the determinant,

$$\begin{vmatrix} \lambda a & \lambda b \\ c & d \end{vmatrix} = \lambda \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

<u>Remark</u>: Note that this implies that

$$\begin{vmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{vmatrix} = \lambda^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and in general,}$$

$$\| \det(\lambda A) = \lambda^n \det(A) \|$$ (important and source of typical mistakes).

3.2) Addition in one row is linear:

$$\begin{vmatrix} a+\tilde{a} & b+\tilde{b} \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \tilde{a} & \tilde{b} \\ c & d \end{vmatrix}$$

· <u>Remark</u>: Note that we are <u>defining</u> the determinant by its properties. You can "check" the last one (for example) using the "formula"

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

but this formula is <u>derived</u> from the ~~form~~ properties.

• <u>Remark</u>: The previous "definition" is indeed a definition because it can be proved that such a function is unique (not too difficult to prove, not too interesting either: google).
(or office hours...)

Right now we "don't know" how to compute determinants yet for a particular matrix.
Let's derive some useful properties (both for theoretical and computing reasons):

4) If two rows are equal, then the determinant is zero.

Let $\tilde{A}$ be the matrix A with the two equal rows exchanged. Then

$\det \tilde{A} = -\det A$. But clearly $\tilde{A} = A$, so

$\det A = -\det A \Rightarrow 2\det A = 0 \Rightarrow \det A = 0$.

5) Elimination steps don't change the det:

$$\begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix} \underset{\text{prop. 3.2)}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -\ell a & -\ell b \end{vmatrix} \underset{\text{prop. 4)}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \ell \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0}$$

6) If A has a row of zeros, the $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \underset{\text{prop. 5)}}{=} \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

(same proofs for $n\times n$ matrices).

7) Prop. 5) and 6) $\Rightarrow$ If a matrix has some linearly dependent rows, then its det is zero.

8) For triangular matrices, the determinant is equal to the product of its diagonal entries.

If the diagonal entries are not zero, then by elimination one obtains

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} \underset{\text{prop. 5)}}{=} \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} \underset{\text{prop. 3.i)}}{=} a_{11}\begin{vmatrix} 1 & 0 \\ 0 & a_{22} \end{vmatrix} \underset{\text{prop. 3.i)}}{=} a_{11}a_{22}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \underset{\text{prop. 1)}}{=} a_{11}a_{22}$$

If a diagonal entry is zero, elimination on the rest of columns will produce a row of zeros:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}\right) \underset{\text{prop. 5)}}{=} \begin{vmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \underset{\text{prop. 6)}}{=} 0$$

9) $\det(A) = \pm$ product of pivots (we allow ~~these~~ pivots to be zero here).

↳ By elimination steps and some possible row exchanges (which give the $\pm$) we go from A to U (upper triangular with diagonal entries = pivots).

<u>Example</u>:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

<u>Remark</u>: This is the way computers calculate $\det(A)$.
It is the most efficient way for big matrices.

10) A is invertible $\iff \det(A) \neq 0$

[Recall that A was invertible if we had n pivots ($\neq 0$)]

This the most important property for us.

11) $\det(AB) = \det(A)\det(B)$ [Very useful]

(A, B square matrices) (Proof: in the book / if interested).
google / office hours

12) $\det(A^T) = \det(A)$

↳ All the previous properties can now be stated in terms of columns instead of rows.

(i.e., if a column is all zeros, the determinant is zero; if some column are lin. dep., the det is zero; ...).

13) $\det(A) \neq 0 \iff$ columns of A are linearly independent.
(rows)

(We can also see this through pivots)

14) $\det(A^{-1}) = \dfrac{1}{\det(A)}$ (if A invertible)

This is a consequence of 11):

$$\left.\begin{aligned}\det(AA^{-1}) &= \det(I) = 1\\ &= \det(A)\det(A^{-1})\end{aligned}\right\} \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

• Example:

1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det(A)$?

Sol: Assume $a \neq 0$. Then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = ad - cb.$$

If $a = 0$,

$$\begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ 0 & b \end{vmatrix} = -cb.$$

So, in any case, $\| \det(A) = ad - cb \|$.

2) We know that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is

(not done)

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{when } \det(A) \neq 0).$$

Check that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Sol:

$$\det(A^{-1}) = \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{(ad-bc)^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix} =$$

$$= \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} \,/\!/.$$

• <u>Exercise</u>: Find the error:

Let $CD = -DC$. Then, $|C||D| = -|D||C|$, thus

$2|C||D| = 0$ and therefore $|C| = 0$ or $|D| = 0$.

Sol: Error is that $CD = -DC \not\Rightarrow |C||D| = -|D||C|$ in general.

Recall that

$$\det(\lambda A) = \lambda^n \det(A), \text{ so } \det(-A) = (-1)^n \det(A).$$

In this case, we would obtain

$$\det(CD) = \det(-DC) \Rightarrow |C||D| = (-1)^n |D||C|.$$

So if n is even, $|C|$ and $|D|$ can be different than zero.

• <u>Exercise</u>: Find the error

(not done)

$$P = A(A^T A)^{-1} A^T \Rightarrow |P| = |A| \frac{1}{|A^T||A|} |A^T| = 1.$$

<u>Sol</u>: The matrix A is not square in most cases, so it is meaningless to take determinants.

• <u>Exercise</u>: True / False

1) If A is not invertible, the AB is not invertible.

True: $\det(AB) = \det(A)\det(B) = 0 \Rightarrow AB$ not invertible.

$\uparrow$
A not invertible
$\Leftrightarrow \det(A) = 0$

2) $\det(A)$ is always equal to the product of its pivots.

False: $\det(A) = \underset{\uparrow}{\pm}$ product pivots.

3) $\det(A+B) = \det(A) + \det(B)$

False: $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4 \neq \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$