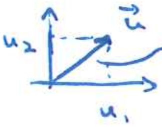
MATH 312 LECTURE 11 : Orthogonality and Projections.

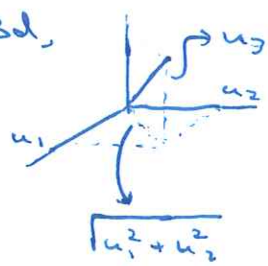
(Review/finish matrix of lin. transf.)(pages -85- to -87-).

I.) Orthogonal complements

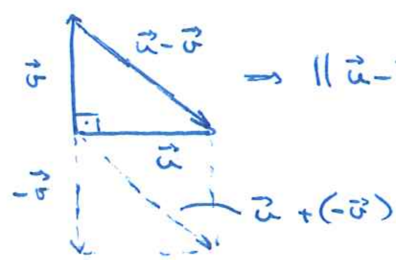
Length of a vector: $\|\vec{u}\| = (u_1^2 + u_2^2 + \dots + u_n^2)^{1/2} = (\vec{u}\cdot\vec{u})^{1/2} = (\vec{u}^T\vec{u})^{1/2}$

Pythagora's th. : 

$(\text{length})^2 = u_1^2 + u_2^2$

In 3d, 

So $(\text{length})^2 = u_3^2 + \left(\sqrt{u_1^2+u_2^2}\right)^2 = u_1^2 + u_2^2 + u_3^2$.

· Orthogonal vectors: 

$\Rightarrow \|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \Rightarrow$

$$\left.\begin{aligned} \Rightarrow \underbrace{u_1^2 + \dots + u_n^2 + v_1^2 + \dots + v_n^2}_{\|\vec{u}\|^2 + \|\vec{v}\|^2} &= \underbrace{(u_1 - v_1)^2 + \dots + (u_n - v_n)^2}_{\|\vec{u}-\vec{v}\|^2} = \\ &= u_1^2 + v_1^2 - 2u_1v_1 + \dots + u_n^2 + v_n^2 - 2u_nv_n \end{aligned}\right\} \Rightarrow$$

$\Rightarrow -2u_1v_1 - 2u_2v_2 - \dots - 2u_nv_n = 0$, i.e., $\vec{u}\cdot\vec{v} = 0$!

So, two vectors $\vec{u}, \vec{v}$ are orthogonal (or perpendicular) if and only if $\vec{u}\cdot\vec{v} = 0$.

• <u>Remark</u>: Recall that $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = \vec{v}^T\vec{u}$

$$\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \dots + u_nv_n.$$

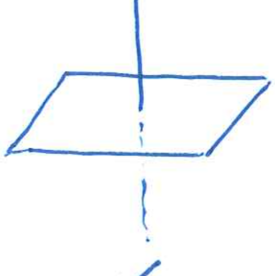
• <u>Remark</u>: The zero vector, $\vec{0}$, is orthogonal to any vector.

• <u>Def</u>: Two subspaces U and V of $\mathbb{R}^n$ are orthogonal if every vector $\vec{u} \in U$ is orthogonal to every vector $\vec{v} \in V$, i.e.,

$U \perp V$ if $\forall \vec{u} \in U, \forall \vec{v} \in V$, then $\vec{u}\cdot\vec{v} = 0$.

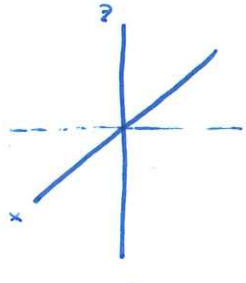
Examples:

a)



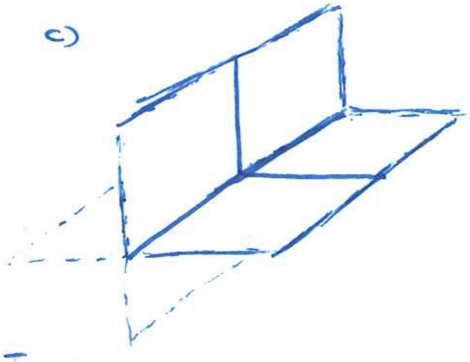
✓

b)



OX, OZ ✓

c)



Two intersecting planes ✗.

• **Def:** Orthogonal complement

Let V be a subspace of $\mathbb{R}^n$. Then, the collection of all vectors orthogonal to V is also a subspace of $\mathbb{R}^n$, denoted by $V^\perp$ (and called "the orthogonal complement" of V).

Examples: a) ✓, b) ×, c) ×

they are orthogonal, but they don't "fill" $\mathbb{R}^n$.

→ The definition requires:

1) V and V^T are orthogonal ($\forall \vec{a} \in V, \vec{b} \in V^T$, $\vec{a} \cdot \vec{b} = 0$).

2) $\dim V + \dim V^T = n$ $\quad (V, V^T \leq \mathbb{R}^n)$

- <u>Fundamental Theorem of Linear Algebra</u>

Let A be an $m \times n$ matrix. Then,

$$\left.\begin{array}{l} \text{1) } N(A) \text{ and } C(A^T) \text{ are subspaces of } \mathbb{R}^n, \\ \text{2) } N(A^T) \text{ and } C(A) \text{ are subspaces of } \mathbb{R}^m, \\ \dim N(A) + \dim C(A^T) = n, \\ \dim N(A^T) + \dim C(A) = m, \\ \dim C(A) = \dim C(A^T) = \operatorname{rank}(A). \end{array}\right\} \leftarrow \text{(we knew all this)} \quad \text{and}$$

$$\left\| \begin{array}{l} N(A) = C(A^T)^\perp \\ C(A) = N(A^T)^\perp \end{array} \right\|$$

- Let's see why the second part is true:

1) $\underline{N(A) = C(A^T)^\perp}$:

$N(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$, $C(A^T) = \operatorname{span}\{\text{rows of } A\}$.

So, if $\vec{x} \in N(A)$, then

$$A\vec{x} = \vec{0} \rightsquigarrow \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ that is,}$$

$$\left.\begin{array}{l} (\text{row } 1) \cdot \vec{x} = 0 \\ \quad \vdots \\ (\text{row } m) \cdot \vec{x} = 0 \end{array}\right\} \Rightarrow \vec{x} \text{ is orthogonal to } \underline{\underline{\text{all}}} \text{ rows of } A,$$

therefore, $\vec{x}$ is orthogonal to all vectors in $C(A^T)$, as they can be written as linear combinations of the rows:

$$\vec{b} \in C(A^T) \rightarrow \vec{b} = \alpha_1 (\text{row } 1) + \dots + a_m (\text{row } m) \rightarrow$$

$$\Rightarrow \vec{x} \cdot \vec{b} = \alpha_1 \underbrace{\vec{x} \cdot (\text{row } 1)}_{=0} + \dots + \alpha_m \underbrace{\vec{x} \cdot (\text{row } m)}_{=0} = 0 \,/\!/.$$

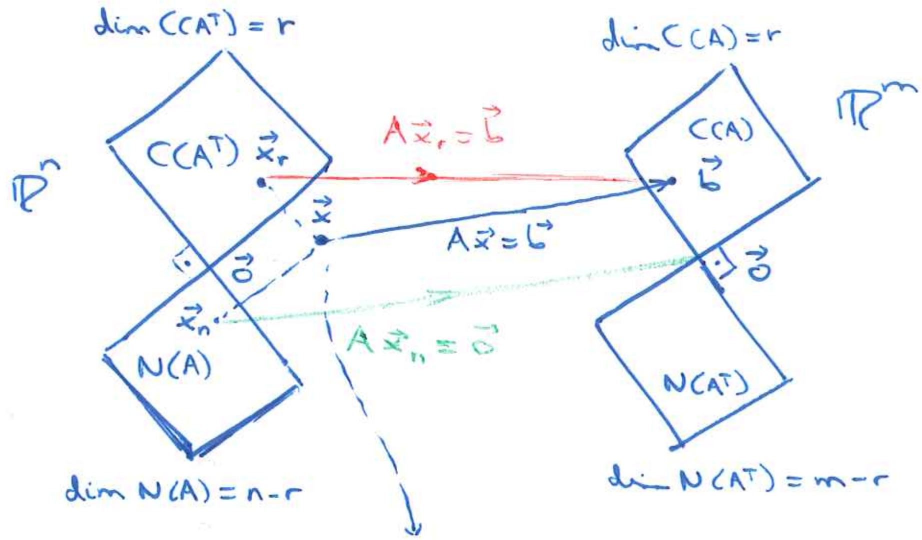
Finally, we know that $\dim N(A) + \dim C(A^T) = n$.

That is,

$$\left.\begin{array}{l} N(A) \perp C(A^T) \\ \dim N(A) + \dim C(A^T) = n \end{array}\right\} \Rightarrow N(A) \text{ and } C(A^T) \text{ are orthogonal complements} /\!/.$$

2) $\underline{C(A) = N(A^T)^{\perp}}$ → Do at home as an exercise ‖

Full picture



$$\begin{cases} \vec{x} = \vec{x}_r + \vec{x}_n \\ \vec{x}_r \perp \vec{x}_n \end{cases}$$

- Given any $\vec{x} \in \mathbb{R}^n$, we can decompose it as

$$\underbrace{\vec{x}}_{} = \underbrace{\vec{x}_r}_{\in C(A^T)} + \underbrace{\vec{x}_n}_{\in N(A)} \quad \text{with } \vec{x}_r \cdot \vec{x}_n = 0 .$$

→ Remark: Given $\vec{b} \in C(A)$, there might be many $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

However, there is only one $\vec{x} \in C(A^T)$ such that $A\vec{x} = \vec{b}$.

That is, all the $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$, are of the form $\vec{x} = \underbrace{\vec{x}_r}_{\in C(A^T)} +$ "anything in $N(A)$".

→ This can be use to define an "inverse" from $C(A)$ to $C(A^T)$.
↳ pseudoinverse.