MIDTERM EXAM 2

MATH 312, GROUP 001

Version A

Name:

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8×11 cheat-sheet.

Problem Number	Possible Points	Points Earned
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1

[20 points]

Part a. [7 points] Apply Gram-Schmidt algorithm to find an orthonormal basis $\{q_1, q_2, q_3\}$ with the same span as

$$\{a_1, a_2, a_3\} = \{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \}.$$

$\vec{v}_1 = \vec{a}_1$

$$\vec{v}_2 = \vec{a}_2 - \left(\frac{\vec{v}_1 \cdot \vec{a}_2}{\|\vec{v}_1\|^2}\right)\vec{v}_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$$

$$\vec{v}_3 = \vec{a}_3 - \left(\frac{\vec{v}_1 \cdot \vec{a}_3}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{a}_3}{\|\vec{v}_2\|^2}\right)\vec{v}_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \cancel{\frac{0}{\|\vec{v}_2\|^2}}\vec{v}_2 = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

We can check $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$

Finally,

$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{q}_2 = \vec{v}_2$, $\vec{q}_3 = \vec{v}_3$ as $\vec{v}_2, \vec{v}_3$ are unitvectors.

So $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\} = \{ \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} \}$ is an orthonormal basis with same span as $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

Part b. We want to find the coefficients a, b, c of a model $z = a + bx + cy$. We are given the following data points: $x = 1, -1, -1, 1$, $y = 1, -1, 1, -1$ and $z = 1, 2, 3, 4$.

1. **Part b.1.** [5 points] Write down in *matrix form* the four equations that would be satisfied if the plane went through all 4 points. What is special about the columns of the matrix?

2. **Part b.2.** [8 points] Find the coefficients a, b, c that minimize the least square error.

1) $$\left.\begin{aligned} 1 &= a+b+c \\ 2 &= a-b-c \\ 3 &= a-b+c \\ 4 &= a+b-c \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{b}}$$

Note that the columns of A are orthogonal. Indeed, from part a),

$A = 2Q$ with Q orthogonal.

2) $A\vec{x} = \vec{b} \rightarrow$ Normal equations $A^T A \vec{x} = A^T \vec{b} \Rightarrow$

$$\Rightarrow 4 \underbrace{Q^T Q}_{=I} \vec{x} = A^T \vec{b} \Rightarrow \vec{x} = \frac{1}{4} A^T \vec{b} =$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 10 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 0 \\ -1/2 \end{bmatrix}$$

Problem 2

[20 points] Let $V = \{(x,y,z) \in \mathbb{R}^3 : x + 2y + z = 0\}$. Consider the following linear transformation T: projection of vectors in $\mathbb{R}^3$ onto V.

Part a. [5 points] If P is the usual 3 by 3 projection matrix (i.e., the matrix of the linear transformation T using the canonical basis), find three eigenvalues and three independent eigenvectors of P. (Hint: No need to compute P).

Note that $\left| \begin{array}{l} P\vec{x} = \vec{x} \text{ for any } \vec{x} \in V \text{ (anything in the plane is projected to itself)} \\ P\vec{x} = \vec{0} \text{ for any } \vec{x} \in V^{\perp} \text{ (vectors perpendicular to V are sent to zero).} \end{array} \right.$

$$\left.\begin{array}{l} \text{Basis of } V = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} = \{\vec{a}_1, \vec{a}_2\} \\ \text{Basis for } V^{\perp} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \{\vec{a}_3\} \end{array}\right\} \Rightarrow \begin{array}{l} \vec{a}_1, \vec{a}_2 \text{ are eigenvectors with } \lambda_1 = \lambda_2 = 1. \\ \vec{a}_3 \text{ is eigenvector with } \lambda_3 = 0. \end{array}$$

Part b. [5 points] Show how to write the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis

$$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\},$$

that is, how to find $\mathbf{b}|_{\mathcal{U}}$ (Do not waste time performing computations).

Let $\mathcal{E}$ denote the standard basis. We have that $\vec{b}|_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Let $\vec{b}|_{\mathcal{U}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$; that is,

$\vec{b} = a\vec{u}_1 + b\vec{u}_2 + b\vec{u}_3$. It is thus clear that

$$\vec{b}|_{\mathcal{E}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \vec{b}|_{\mathcal{U}} \Rightarrow \vec{b}|_{\mathcal{U}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \overset{(*)}{=} \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

• **Remark**: $\mathcal{U}$ is not orthonormal basis, so $\vec{b} \neq (\vec{b}\cdot\vec{u}_1)\vec{u}_1 + (\vec{b}\cdot\vec{u}_2)\vec{u}_2 + \ldots$ ‖.

(*) Not needed
(If you want to compute, solve the system, not the inverse!)

Part c. [5 points] Find an orthonormal basis $\mathcal{V}$ for V.

$$\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{g}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad \left(\text{Check: } \vec{g}_1\cdot\vec{g}_2 = 0\right).$$

$$\vec{v}_2 = \vec{a}_2 - \left(\frac{\vec{a}_2\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ -1 \end{bmatrix} \Rightarrow \vec{g}_2 = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$$

$\mathcal{V} = \{\vec{g}_1, \vec{g}_2\}$ (*) different answers are possible: it depends on your choise for $\vec{a}_1, \vec{a}_2$).

Part c. [5 points] Find the matrix of the linear transformation T when the input basis is $\mathcal{U}$ and the output basis is $\mathcal{V}$.

$$M = \begin{bmatrix} T(\vec{u}_1)|_{\mathcal{V}} & T(\vec{u}_2)|_{\mathcal{V}} & T(\vec{u}_3)|_{\mathcal{V}} \end{bmatrix}$$

$\mathcal{V}$ is an orthonormal basis, so

$$T(\vec{u}_1) = (\vec{u}_1\cdot\vec{g}_1)\vec{g}_1 + (\vec{u}_1\cdot\vec{g}_2)\vec{g}_2 \rightarrow T(\vec{u}_1)|_{\mathcal{V}} = \begin{bmatrix} \vec{u}_1\cdot\vec{g}_1 \\ \vec{u}_2\cdot\vec{g}_2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 6/\sqrt{30} \end{bmatrix}$$

$$T(\vec{u}_2) = \dots \Rightarrow T(\vec{u}_2)|_{\mathcal{V}} = \begin{bmatrix} \vec{u}_2\cdot\vec{g}_1 \\ \vec{u}_2\cdot\vec{g}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$ (of course: you can just look at $\vec{u}_2$ and $\vec{g}_1$!)

$$T(\vec{u}_3)|_{\mathcal{V}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ($\vec{u}_3$ is perpendicular to V, so its projection is zero).

$$M = \begin{bmatrix} 2/\sqrt{5} & \sqrt{5} & 0 \\ 6/\sqrt{30} & 0 & 0 \end{bmatrix} \quad (\underline{\underline{2\times 3}} \text{ matrix})$$ [*) the answer depends on your choise for $\mathcal{V}$].

PROBLEM 3

[20 points] Consider the linear differential system

$$x' = 3x - 4y$$
$$y' = 2x - 3y.$$

Part a. [2 points] For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$$

Part b. [4 points] Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.

$$(3-\lambda)(-3-\lambda)+8=0 \Rightarrow \lambda^2-9+8=0 \Rightarrow \lambda_1=1,\ \lambda_2=-1.$$

$$A-I = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ A+I = \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Part c. [4 points] Find the exponential matrix e^{At}.

$$e^{At} = S\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2e^t - e^{-t} & -2e^t + 2e^{-t} \\ e^t - e^{-t} & -e^t + 2e^{-t} \end{bmatrix}$$

Part d. [5 points] Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = 1$ and $y(0) = 0$. What is the ratio $y(t)/x(t)$ as t goes to infinity?

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{tA}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} x(0) - y(0) \\ -x(0) + 2y(0) \end{bmatrix} =$$

$$= e^{t}(x(0) - y(0))\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-t}(-x(0) + 2y(0))\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{For } x(0) = 1,\ y(0) = 0$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ ; \ \lim_{t \to +\infty} \frac{y(t)}{x(t)} = \frac{1}{2} \leftarrow \text{slope of the "winning" eigenvector.}$$

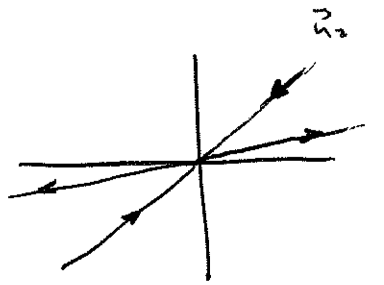
Part e. [5 points] For what initial conditions $x(0)$, $y(0)$ does the solution $(x(t), y(t))$ to this differential system lie on a single straight line in $\mathbb{R}^2$ for all t? (Hint: You can do this explicitly, as in d), or just thinking on the phase portrait)

From d.), we see that

$\to$ If $x(0) = y(0)$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = y(0)e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to$ so it is on a line

$\to$ If $x(0) = 2y(0)$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = y(0)e^{t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \to$ so it is on a line.

This correspond to the two lines given by the eigenvectors:



Any initial data on these two lines make the solution to be on those lines forever.

PROBLEM 4

[20 points]

Consider the ellipse $4x^2 + 6xy + 4y^2 = 1$.

Part a. [10 points] Find M such that $\begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} = 1$. Is M positive definite? Diagonalize M as $M = QDQ^T$ where Q is an orthogonal matrix and D is diagonal.

$$M = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \to (4-\lambda)^2 = 9 \to \lambda = 4 \pm 3 \begin{cases} \lambda_1 = 7 \\ \lambda_2 = 1 \end{cases} \Rightarrow M \text{ is positive definite.}$$

$$M - 7I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M - I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

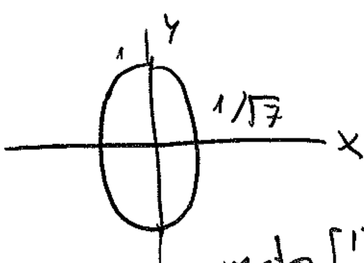
$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

Part b. [10 points] Find the principal axis of the ellipse and the lengths of its semiaxis. Sketch the ellipse in the original x-y plane.

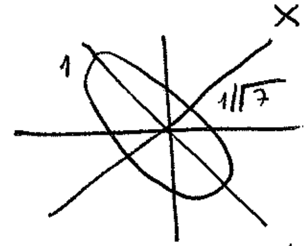
$$\vec{x}^T M \vec{x} = \vec{x}^T Q D Q^T \vec{x} = 1 \Rightarrow \lambda_1 X^2 + \lambda_2 Y^2 = 1 \Rightarrow$$

for $\vec{X} = \begin{bmatrix} X \\ Y \end{bmatrix} = Q^T \vec{x}$

$$\to \left(\frac{X}{1/\sqrt{\lambda_1}}\right)^2 + \left(\frac{Y}{1/\sqrt{\lambda_2}}\right)^2 = 1 \Rightarrow \text{Lengths of semiaxis } \frac{1}{\sqrt{7}}, 1.$$



$\vec{x} = Q\vec{X}$



$X \equiv \vec{q}_1$ (line $x=y$)

$Y \equiv \vec{q}_2$ (line $x=-y$)

vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes to $\vec{q}_1$

" $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ goes to $\vec{q}_2$

Problem 5

[20 points: 2.5 points each] In each of the following cases, clearly mark the statement as **true** or **false**. Please also explain your answers in order to receive credit for this problem.

1. The transformation $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(\mathrm{x}) = x_1 - x_2$ is linear (here $\mathrm{x} = (x_1, x_2)$).

True: $T(\vec{x}) = [1\ -1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ✓.

We can also check definition: 1) $T(\vec{a}+\vec{b}) = T\left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}\right) =$

$= a_1+b_1-a_2-b_2 = T(\vec{a}) + T(\vec{b})$ //.

2) $T(\lambda\vec{a}) = T\left(\begin{bmatrix} \lambda a_1 \\ \lambda a_2 \end{bmatrix}\right) = \lambda a_1 - \lambda a_2 = \lambda(a_1 - a_2) = \lambda T(\vec{a})$ //

2. A definite positive matrix always has an inverse.

True: A positive def. $\Rightarrow$ All $\lambda > 0 \Rightarrow \det(A) > 0$ //.

(in particular, $\det(A) \neq 0$)

3. If A is a 2 by 2 matrix with eigenvalues -1 and 2, and B is a 2 by 2 matrix with eigenvalues 0 and 1, then $\det((B-I)^2 A) = 1$.

False: B has $\lambda_1 = 0, \lambda_2 = 1 \Rightarrow B - I$ has eigenvalues $-1, 0 \Rightarrow$

$\Rightarrow \det(B-I) = 0 \Rightarrow \det((B-I)^2 A) = (\det(B-I))^2 \det(A) = 0$.

4. It is possible to find a matrix A such that the matrix AA^T have $\lambda = -1$ as an eigenvalue.

False: AA^T is always positive semidefinite,

(*)

$\vec{x}^T A A^T \vec{x} = (A^T\vec{x})^T (A^T\vec{x}) = \|A^T\vec{x}\|^2 \geq 0$ //.

I am assuming A is real.

5. If A is an orthogonal matrix, then $\lambda = 2$ cannot be an eigenvalue.

(you can prove that, but not needed)

True: A orthogonal $\Rightarrow$ all λ's verify $|\lambda| = 1$;

$A\vec{x} = \lambda\vec{x} \Rightarrow \|A\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\|\vec{x}\| \Rightarrow |\lambda| = 1$.

Since it is not stated, complex counterexamples are valid.

$\|A\vec{x}\| = \|\vec{x}\|$ for A orthogonal

6. For any square matrice A, $\det(2A) = 2\det(A)$.

False: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(2I) = 4 \neq 2 = 2\det(I)$

In general, $\det(2A) = 2^n \det(A)$ for A $n \times n$.

7. Let A be an n by n matrix. If n is odd and A is skew-symmetric (i.e., $A^T = -A$), then A is not invertible.

True: $|A^T| = |A|$ always.

$$\left.\begin{array}{l} \text{Since } A^T = -A \Rightarrow |A^T| = |-A| = (-1)^n |A| \overset{n \text{ odd}}{=} -|A| \end{array}\right\} \Rightarrow$$

$\Rightarrow |A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0 \Rightarrow A$ is not invertible.

8. For any matrix A, the nullspace of A is exactly the same as the nullspace of $A^T A$.

True:

1) $\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow A^T A \vec{x} = A^T \vec{0} = \vec{0} \Rightarrow \vec{x} \in N(A^T A)$.

2) $\vec{x} \in N(A^T A) \Rightarrow A^T \underbrace{A\vec{x}} = \vec{0} \Rightarrow A\vec{x} \in N(A^T)$

Recall that $N(A^T) = C(A)^\perp$.

But $A\vec{x} \in C(A)$, since $A\vec{x}$ is just some combinations of the columns of A.

Then,

$$\left.\begin{array}{l} A\vec{x} \in C(A) \\ A\vec{x} \in C(A)^\perp \end{array}\right\} \Rightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} \in N(A) /\!/.$$

Extra space for work:

Extra space for work: