• Review for the ~~exam~~ exam:

1) Least squares → Fit a model
→ Solve a system $A\vec{x}=\vec{b}$ (with no solution) in the least squares sense.

2) Orthonormal vectors: → Matrix with orthonormal columns: $Q^TQ = I$.
→ Matrix with orthonormal rows: $QQ^T = I$.
→ Orthogonal matrix: $Q^TQ = QQ^T = I$.

→ Orthonormal basis → Gram-schmidt.

→ Projection onto orthonormal vectors. Find coordinates on a different orthonormal basis.

3) determinants → properties ($\det(A^n)$, $\det(A^{-1})$, $\det(A \cdot B)$, trace, pivots, ...)
→ area of a triangle.
→ change of volume of corresponding linear transformation.

4) linear transformations → decide if a transformation is linear.
→ find the matrix in given basis.
→ Change of coordinates.

5) Eigenvalues $\longrightarrow$ Solve difference equations $\vec{x}(k+1) = A\vec{x}(k)$.

↑

↳ Diagonalize, e^{At}.

↓

↳ Solve differential equations $\vec{x}'(t) = A\vec{x}(t)$.

↳ Properties ($\sim$ theory)

6) Symmetric matrices and semidefinite positive matrices $\longrightarrow$ Diagonalize as QDQ^T

↳ Principal Axes for ellipses.

↳ Properties ($\sim$ theory)

Extra Problems for Midterm 2

PROBLEM ~~1~~

Suppose you know that the eigenvalues of a 3×3 matrix A are $0, 1, 2$ with corresponding eigenvectors $\vec{v}_0, \vec{v}_1$, and $\vec{v}_2$.

1. What are the eigenvectors and eigenvalues of A^2?

2. What are the eigenvectors and eigenvalues of $A + Id$?

3. Is A invertible? If so what are the eigenvectors and eigenvalues of A^{-1}?

PROBLEM ~~2~~

We want to find the curve $y = a + 2^t b$ that gives the best fit (in the least squares sense) to the data $t = 0, 1, 2,\ y = 6, 4, 0$.

1. Write down the 3 equations that would be satisfied if the curve went though all 3 points.

2. Find the coefficients a, b of the curve of best fit $y = a + 2^t b$.

PROBLEM ~~3~~

A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

1. Apply the Gram-Schmidt process to find two orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$ which also span V.

2. Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.

3. Find the best possible (i.e., least squared error) solution to the linear system

$$Q \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Problem 4

Consider the planes $H = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and $V = \{(x, y, z) \in \mathbb{R}^3 : z = y\}$. We will use the following basis for H:

$$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

1. Write the vector $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ in the basis $\mathcal{E}$. That is, give the coordinates of $\mathbf{u}$ in $\mathcal{E}$.
2. Find an orthonormal basis $\mathcal{V}$ for V.
3. Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis $\mathcal{V}$.
4. The projection of vectors in V onto H is a linear transformation. Find the matrix of this linear transformation using the basis $\mathcal{V}$ and $\mathcal{E}$.
5. Find the area of the triangle with vertex $(0, 0, 0)$, $(1, 1, 1)$ and $(0, 1, 1)$.

Problem 5

Consider the following basis of $\mathbb{R}^3$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \}$, where the coordinates of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are given with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

1. Given a general vector $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ written in the standard basis (i.e., $\mathbf{u} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3$), find its coordinates in $\mathcal{B}$

Problem 6

The polynomials $\vec{u}_1 = 1$, $\vec{u}_2 = x - 2$, and $\vec{u}_3 = (x - 2)^2$ form a basis for the space of (at most) quadratic polynomials in x, as do the polynomials $\vec{v}_1 = 1$, $\vec{v}_2 = x + 1$, and $\vec{v}_3 = (x + 1)^2$. Find the change of basis matrix from $\{u_i\}$ to $\{v_i\}$ and use it to find numbers a, b, c such that $-1(x - 2) + 3(x - 2)^2 = a + b(x + 1) + c(x + 1)^2$.

Problem 7

Find the limit of A^k as k goes to infinity for

$$A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}$$

Problem 8

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & b \end{bmatrix}$$

For which values of b does A have distinct eigenvalues?

Problem 9

Let P be a matrix that projects vectors of $\mathbb{R}^3$ onto the plane $z = 0$. What are the eigenvalues and eigenvectors of P?

Problem 10

Consider the linear differential system

$$\begin{aligned} x' &= x + 3y \\ y' &= 2x + 2y. \end{aligned}$$

1. For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?
2. Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.
3. Write the exponential matrix e^{At}
4. Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = -5$ and $y(0) = 5$.
5. If $x(t)$ and $y(t)$ represent two species that have a mutually symbiotic relationship, say $x(t)$ number of flowers and $y(t)$ number of bees, how many bees per flowers are there in the equilibrium situation (that is, as time goes to infinity)?

Problem 11

Find an orthogonal matrix Q and a diagonal matrix D such that $M = QDQ^T$, where

$$M = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Is M positive definite?

Problem 12

Given the ellipse $3x^2 + 4xy + 2y^2 = 1$,

1. Find M such that $\begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} = 1$.
2. Find the principal axis of the ellipse and the lengths of its semiaxis. Sketch the ellipse.

Problem 13

Which of the following matrices are positive semi-definite?

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

True or False

1. If A is a 3x3 matrix with determinant 1, then $2A$ has determinant 6.
2. If A is a square matrix and B is obtained from A via row operation R2'=R2+3R1, then B has the same eigenvalues as A.
3. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ corresponding to distinct eigenvalues, then $\mathbf{u}_1^T \mathbf{u}_2 = 0$.
4. A definite positive matrix always has an inverse.
5. If A is invertible and has one eigenvalue λ, then $1/\lambda$ is an eigenvalue of A^{-1}.
6. Is the following transformation linear? $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(\mathbf{x}_0) =$ solution to the system given by $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \mathbf{x}$ with initial condition $\mathbf{x}_0$.
7. Is the following transformation linear? $$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \mathbf{x} &\to T(\mathbf{x}) = \min_{i=1,2,3} x_i \end{aligned}.$$
8. If Q is an orthogonal matrix, then the corresponding linear transformation preserves lenghts and angles, i.e., length of $Q\mathbf{x}$ is equal to length of $\mathbf{x}$ and the angle between $\mathbf{x}$ and $\mathbf{y}$ is equal to the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$.
9. An square matrix with orthonormal columns always has orthonormal rows.
10. A 3 by 3 symmetric matrix with eigenvalues $0, 0, 1$ always has rank(A)=1.

11. A 2 by 2 matrix that rotates every vector 90° cannot have any real eigenvalues.

12. The matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$ has an eigenvalue equal to 9.

13. The matrix $A^T A$ is always positive semidefinite.

14. A basis for eigenvectors for nonzero eigenvalues of A is a basis for $C(A)$ for any matrix A.

15. The eigenvectors for the zero eigenvalue are the null space of A.

16. The only upper triangular 3×3 matrix with 1s on the diagonal which is diagonalizable is the identity matrix.

$\boxed{1}$

1) $A\vec{v}_0 = \vec{0}$
$A\vec{v}_1 = \vec{v}_1$
$A\vec{v}_2 = 2\vec{v}_2$

$\Big\}$ For A^2 we have $0, 1, 4$ and same eigenvectors $\vec{v}_0, \vec{v}_1, \vec{v}_2$.

↳ $A\vec{v}_i = \lambda_i \vec{v}_i$

$A^2 \vec{v}_i = \lambda_i A\vec{v}_i = \lambda_i^2 \vec{v}_i$.

2) For $A + I \to 1, 2, 3$ with same eigenvectors.

↳ $A\vec{v}_i = \lambda \vec{v}_i \Rightarrow (A + I)\vec{v}_i = A\vec{v}_i + \vec{v}_i = \lambda_i \vec{v}_i + \vec{v}_i = (\lambda_i + 1)\vec{v}_i$.

3) If $\lambda_i, \vec{v}_i$ eigenvalue and corresponding eigenvector of A,

then $A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow A^{-1}A\vec{v}_i = \lambda_i A^{-1}\vec{v}_i \Rightarrow A^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i \Rightarrow$

$\Rightarrow \frac{1}{\lambda_i}$ is eigenvalue of A^{-1} with same eigenvector.

$\boxed{1}$

1) $\left.\begin{aligned} 6 &= a + b \\ 4 &= a + 2b \\ 0 &= a + 4b \end{aligned}\right\}$ System with no solution.

2) $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}}_{\vec{b}} \rightarrow$ Solve $A\vec{x} = \vec{b}$ in the least squares sense.

Normal equations: $A^T A \vec{x} = A^T \vec{b}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{63-49} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 210 - 98 \\ -70 + 42 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 112 \\ -28 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\boxed{3}$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

1) $\vec{v}_1 = \vec{a}_1$

$$\vec{v}_2 = \vec{a}_2 - \left(\frac{\vec{v}_1 \cdot \vec{a}_2}{\|\vec{v}_1\|^2}\right)\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} \to \|\vec{v}_2\|^2 = \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$$

So
$$\vec{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\sqrt{2/3}}\begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

2) In general, $A(A^TA)^{-1}A^T$, where A has the basis vectors in the cols.

Here, $P = Q(Q^TQ)^{-1}Q^T = QQ^T$ with
$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & \sqrt{3}/3\sqrt{2} \\ -1/\sqrt{3} & 2\sqrt{3}/3\sqrt{2} \\ 1/\sqrt{3} & \sqrt{3}/3\sqrt{2} \end{bmatrix} =$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1/\sqrt{2} \\ -1 & 2/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{bmatrix},$$

so
$$QQ^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1/\sqrt{2} \\ -1 & 2/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{bmatrix} \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 & 1 \\ 1/\sqrt{2} & 2/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3/2 & 0 & 3/2 \\ 0 & 3 & 0 \\ 3/2 & 0 & 3/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

3) $Q\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightsquigarrow \underbrace{Q^T Q}_{= I}\begin{bmatrix} x \\ y \end{bmatrix} = Q^T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightsquigarrow$

$\rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 & 1 \\ 1/\sqrt{2} & 2/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 \\ 5/\sqrt{2} \end{bmatrix}$

$\boxed{4}$

1) $\vec{u} = 2\vec{e}_1 - \vec{e}_2 \rightsquigarrow \vec{u}\big|_{\mathcal{E}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2) $V \equiv z = y \rightsquigarrow \left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

So we have $V = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\right\} = \{\vec{v}_1, \vec{v}_2\}$

3) $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (\vec{v}\cdot\vec{v}_1)\vec{v}_1 + (\vec{v}\cdot\vec{v}_2)\vec{v}_2 = 1\vec{v}_1 + \frac{2}{\sqrt{2}}\vec{v}_2 \Rightarrow \vec{v}\big|_{V} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$

4) $M = \left[\, T(\vec{v}_1) \;\middle|\; T(\vec{v}_2) \,\right]$

projection of $\vec{v}_1, \vec{v}_2$ onto H (write in $\mathcal{E}$ basis)

$T(\vec{v}_1) = (\vec{v}_1\cdot\vec{e}_1)\vec{e}_1 + (\vec{v}_1\cdot\vec{e}_2)\vec{e}_2 = 1\vec{e}_1 + 0\vec{e}_2 \rightsquigarrow T(\vec{v}_1)\big|_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$T(\vec{v}_2) = (\vec{v}_2\cdot\vec{e}_1)\vec{e}_1 + (\vec{v}_2\cdot\vec{e}_2)\vec{e}_2 = 0\vec{e}_1 + \frac{1}{\sqrt{2}}\vec{e}_2 \Rightarrow T(\vec{v}_2)\big|_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \end{bmatrix}$

So $M = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$

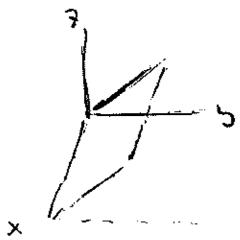
5) Notice that this lies on V:

$$\vec{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 \Rightarrow \vec{a}|_V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \vec{v}_1 + \sqrt{2}\, \vec{v}_2 \Rightarrow \vec{b}|_V = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \vec{v}_1 + \sqrt{2}\, \vec{v}_2 \Rightarrow \vec{c}|_V = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

$$\text{Área} = \left| \frac{1}{2} \begin{vmatrix} 1 & 0 \\ \sqrt{2} & \sqrt{2} \end{vmatrix} \right| = \frac{\sqrt{2}}{2}$$



$\boxed{\text{P\#5}}$

$\vec{u} = a\,\vec{e}_1 + b\,\vec{e}_2 + c\,\vec{e}_3$, $\vec{u}\,|_{\mathcal{B}}$?

$$\begin{aligned} \vec{b}_1 &= \vec{e}_1 + \vec{e}_2 & & \vec{e}_1 = \\ \vec{b}_2 &= -\vec{e}_2 + \vec{e}_3 & \rightsquigarrow \quad & \vec{e}_2 = \quad ? \\ \vec{b}_3 &= 2\vec{e}_1 + 2\vec{e}_2 + \vec{e}_3 & & \vec{e}_3 = \end{aligned}$$

Let $S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. Then given a vector $\vec{v}$ written in basis $\mathcal{B}$, we have that

$\uparrow \quad \uparrow \quad \uparrow$

$\vec{b}_1|_{\mathcal{E}} \quad \vec{b}_2|_{\mathcal{E}} \quad \vec{b}_3|_{\mathcal{E}}$

$$\vec{v}\,|_{\mathcal{E}} = S\,\vec{v}\,|_{\mathcal{B}}$$

Therefore, $\vec{v}\,|_{\mathcal{B}} = S^{-1}\vec{v}\,|_{\mathcal{E}}$

Thus,

$$\vec{u}\,|_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\boxed{P10}$

1) $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \vec{x}(t)$

2) $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \to (1-\lambda)(2-\lambda) - 6 = 0 \Leftrightarrow \lambda^2 - 3\lambda + 2 - 6 = 0$

$$\lambda = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} \begin{cases} \lambda = 4 \\ \lambda = -1 \end{cases}$$

$$\left.\begin{matrix} A - 4I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A + I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \to \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{matrix}\right\} \quad S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

3) $e^{At} = S e^{Dt} S^{-1}, \quad e^{Dt} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix}$

4) $\vec{x}(t) = e^{At}\vec{x}(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

$$= \begin{bmatrix} e^{4t} & -e^{-t} \\ e^{4t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \cancel{5e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \; \frac{1}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\hookrightarrow x(t) = \frac{1}{2} e^{4t} - \frac{1}{2} e^{-t}, \quad y(t) = \frac{1}{2} e^{4t} + \frac{1}{2} e^{-t}$

5) $\lim\limits_{t \to +\infty} \frac{y(t)}{x(t)} = \lim\limits_{t \to +\infty} \frac{\frac{1}{2}e^{4t} + \frac{1}{2}e^{-t}}{\frac{1}{2}e^{4t} - \frac{1}{2}e^{-t}} = 1.$

($\boxed{11}$ in lect. notes)

$\boxed{12}$

1) $M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

2) $(3-\lambda)(3-\lambda)-4=0 \to \lambda^2-6\lambda+9-4=0 \to \lambda = \frac{+6\pm\sqrt{36-20}}{2} = \frac{6\pm 4}{2}$ $\begin{cases} \lambda_1 = 5 \\ \lambda_2 = 1 \end{cases}$

$A - 5I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A - I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So $M = QDQ^T$, with $D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

So the ellipse is

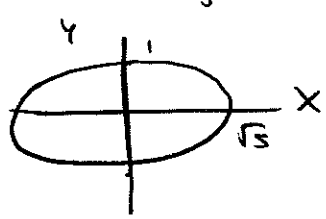
$[x \ \ y]\, QDQ^T \begin{bmatrix} x \\ y \end{bmatrix} = [X \ \ Y]\, D \begin{bmatrix} X \\ Y \end{bmatrix}$ with $\begin{bmatrix} X \\ Y \end{bmatrix} = Q^T \begin{bmatrix} x \\ y \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = Q \begin{bmatrix} X \\ Y \end{bmatrix}$

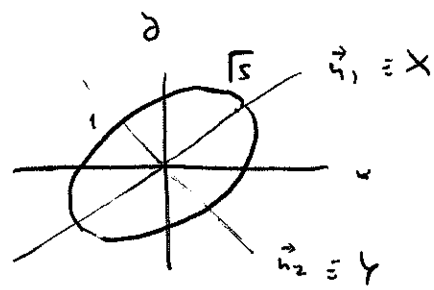
$$= 5X^2 + Y^2 = 1$$

$$\left(\frac{X}{\sqrt{5}}\right)^2 + Y^2 = 1$$

Semiaxis length: $\sqrt{5}$, 1

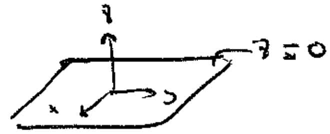


$\xrightarrow{Q}$



[?] [P9]

P projects onto $z=0$



Then $N(P) = z$ axis $\sim \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

So $P\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$, that is, $\lambda = 0$ with $\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigenvalue/eigenvector.

It is also clear that

$$\left. \begin{array}{l} P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ P\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \right\} \Rightarrow \lambda = 1 \text{ repeated, with eigenvectors } \begin{array}{l} \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

(or any other in the xy plane).

$\boxed{PC}$

$B_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{1, x-2, (x-2)^2\}$

$B_2 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{1, x+1, (x+1)^2\}$

Find $M \equiv$ change of basis matrix from B_1 to B_2.

<u>Sol</u>:

1) We want M / $\vec{x}|_{B_2} = M\,\vec{x}|_{B_1}$, that is

$$M = \left[\ \vec{u}_1|_{B_2} \ \vdots\ \vec{u}_2|_{B_2} \ \vdots\ \vec{u}_3|_{B_2}\ \right].$$

$\vec{u}_1 = 1 = \vec{v}_1 \Rightarrow \vec{u}_1|_{B_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{u}_2 = x-2 = -3+x+1 = -3\vec{v}_1 + \vec{v}_2 \Rightarrow \vec{u}_2|_{B_2} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$\vec{u}_3 = (x-2)^2 = x^2 - 4x + 4 = (x+1)^2 - 6x + 3 =$

$= (x+1)^2 - 6(x+1) + 9 = 9\vec{v}_1 - 6\vec{v}_2 + \vec{v}_3 \Rightarrow \vec{u}_3|_{B_2} = \begin{bmatrix} 9 \\ -6 \\ 1 \end{bmatrix}$

So

$$M = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $a, b, c \;/\; -(x-2)+3(x-2)^2 = a+b(x+1)+c(x+1)^2$?

$p(x) = -(x-2)+3(x-2)^2 = a+b(x+1)+c(x+1)^2 \rightarrow$

$\rightarrow p(x)\big|_{B_1} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix},\; p(x)\big|_{B_2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, so

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 30 \\ -19 \\ 3 \end{bmatrix}$$

π Check: $-x+2+3x^2-12x+12 = 30-19x-19+3x^2+3+6x$ ✓

$\boxed{P7}$

$$A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix} \rightarrow \lim_{k\to\infty} A^k \text{ ?}$$

$A-\lambda I \sim \left(\frac{1}{2}-\lambda\right)\left(\frac{1}{4}-\lambda\right)-\frac{3}{24}=0 \Leftrightarrow (1-2\lambda)(1-4\lambda)\frac{1}{\cancel{24}}-\frac{3}{\cancel{24}}=0 \Leftrightarrow$

$\Leftrightarrow 8\lambda^2-6\lambda+1-3=0 \Leftrightarrow \lambda = \dfrac{6\pm\sqrt{36+64}}{16} = \dfrac{6\pm 10}{16}$ $\nearrow \lambda_1 = 1$ $\searrow \lambda_2 = \frac{-1}{4}$

$$A - I = \begin{bmatrix} -1/2 & 3/4 \\ 1/2 & -3/4 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$$A + \frac{1}{4}I = \begin{bmatrix} 3/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \Rightarrow$$

$$\Rightarrow \lim_{k\to\infty} A^k = \begin{bmatrix} 3/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 3/2 \end{bmatrix} \frac{-2}{5} =$$

$$= \begin{bmatrix} 3/2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 3/2 \end{bmatrix} \frac{-2}{5} = \frac{-2}{5} \begin{bmatrix} -3/2 & -3/2 \\ -1 & -1 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 3/2 & 3/2 \\ 1 & 1 \end{bmatrix}$$

$\boxed{P8}$

A is upper triangular $\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = b$.

So if $b \neq 1$ and $b \neq 2$ then A has three distinct eigenvalues.

$\boxed{P13}$

We know that positive semi-definite $\Rightarrow$ all $\lambda \geq 0$.

1) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow (1-\lambda)^2 - 4 = 0 \Leftrightarrow 1-\lambda = \pm 2 \Leftrightarrow \lambda = 1 \pm 2$ $\nearrow \lambda_1 = 3$, $\searrow \lambda_2 = -1$ No

2) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow (1-\lambda)^2 - 1 = 0 \Leftrightarrow 1-\lambda = \pm 1 \Leftrightarrow \lambda = 1 \pm 1$ $\nearrow \lambda_1 = 2$, $\searrow \lambda_2 = 0$ ✓ Yes.

3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \lambda_1 = 0 = \lambda_2 \rightarrow$ ✓ yes.

True / False

1) False $\to$ $\det(2A) = 2^3 \det(A) = 8$.

2) False: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$

$\lambda_1 = 1$, $\lambda_2 = 0$ $\qquad$ $(1-\lambda)(3-\lambda) - 3 = 0$

$\lambda^2 - 4\lambda = 0 \to \lambda_1 = 0,\ \lambda_2 = 4$.

3) True: A is symmetric, so the spectral theorem applies.

4) True: All the eigenvalues are strictly positive, therefore $\det(A) \neq 0 \Rightarrow A^{-1}$ exists.

5) True: $A\vec{x} = \lambda\vec{x} \Rightarrow A^{-1}A\vec{x} = \vec{x} = \lambda A^{-1}\vec{x} \Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$ //.

6) True:

$T(\vec{x}_0) = e^{At}\vec{x}_0$ with $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$. Then,

1) Let $\vec{a}, \vec{b} \in \mathbb{R}^2$: $T(\vec{a} + \vec{b}) = e^{At}(\vec{a} + \vec{b}) = e^{At}\vec{a} + e^{At}\vec{b} = T(\vec{a}) + T(\vec{b})$ //.

2) $\vec{a} \in \mathbb{R}^2$, $c \in \mathbb{R}$: $T(c\vec{a}) = e^{At}c\vec{a} = c e^{At}\vec{a} = c\,T(\vec{a})$ //.

7) False:

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to T(\vec{x}) = \min\{1,1,0\} = 0 .$$

$$T(-\vec{x}) = \min\{-1,-1,0\} = -1 \neq -T(\vec{x}) = 0$$

8) True: $\|Q\vec{x}\|^2 = \vec{x}^T Q^T Q \vec{x} = \vec{x}^T \vec{x} = \|\vec{x}\|^2$

$(Q\vec{x})\cdot(Q\vec{y}) = \vec{x}^T Q^T Q \vec{y} = \vec{x}^T \vec{y} = \vec{x}\cdot\vec{y}$

$= \|Q\vec{x}\|\|Q\vec{y}\|\cos(\theta_1) = \|\vec{x}\|\|\vec{y}\|\cos(\theta_2) \Rightarrow \theta_1 = \theta_2$ //.

9) True: $Q^TQ = I \Rightarrow QQ^T = I$?? (if Q square)

$\left.\begin{array}{l} Q \text{ square} \\ Q^TQ = I \end{array}\right\} \Rightarrow \det(Q^TQ) = I = |Q^T||Q| = |Q|^2 \Rightarrow |Q| = \pm 1 \neq 0 \Rightarrow$

$\Rightarrow Q^{-1}$ exists. Therefore,

$$Q^TQ = I \Rightarrow QQ^TQ = Q \Rightarrow QQ^TQQ^{-1} = QQ^{-1} \Rightarrow QQ^T = I \text{ //.}$$

10) True: Since it is symmetric, we know that $\lambda = 0$ repeated gives two eigenvectors, so $\dim N(A) = 2 \Rightarrow \text{rank}(A) = 1$.

11) True:

By definition, $\vec{x}$ is an eigenvector if it doesn't change direction when multiplied by A:

$$A\vec{x} = \underbrace{\lambda\vec{x}}_{\text{multiple of } \vec{x}}.$$

12) True: rank(A) = 1 → $\lambda = 0$ repeated with 2 distinct eigenvectors.
(N(A) has dim = 2).

So trace(A) = $0 + 0 + \lambda_3$
↳ $\lambda_3 = 1 + 6 + 2 = 9$.

13) True: $\vec{x}^T A^T A \vec{x} = (A\vec{x})^T (A\vec{x}) = \|A\vec{x}\|^2 \geq 0$.

14) False: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has rank(A) = 2, so $C(A) = \mathbb{R}^2$.

However, $\lambda_1 = 1 = \lambda_2$ with only one eigenvector!

15) True: $\lambda = 0 \to A\vec{x} = \vec{0}$ coincides with definition of N(A).

16) True: $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \to \lambda_1 = 1 = \lambda_2 = \lambda_3 \to A - I = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ to have 3 l.i. eigenvectors we need rank(A − I) = 0.