MATH 312
LECTURE 17

: Singular Value Decomposition

From last lecture,

<u>Spectral Theorem</u>: M real and symmetric $\Leftrightarrow$

$$M = QDQ^T, \text{ with } \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$Q = \left[\vec{f}_1 \,\middle|\, \cdots \,\middle|\, \vec{f}_n \right] \text{ orthogonal matrix.}$$

↳ orthonormal eigenvectors.

~~Exam~~

<u>Example</u>: Find $M = QDQ^T$ for $M = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Eigenvalues: $M - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} \rightsquigarrow$

$$\rightsquigarrow (2-\lambda)^3 + 1 + 1 - (2-\lambda) - (2-\lambda) - (2-\lambda) = 0 \Leftrightarrow$$

$$\Leftrightarrow (4 + \lambda^2 - 4\lambda)(2-\lambda) + 2 - 6 + 3\lambda = 0 \Leftrightarrow$$

$$\Leftrightarrow 8 - 4\lambda + 2\lambda^2 - \lambda^3 - 8\lambda + 4\lambda^2 - 4 + 3\lambda = 0 \Leftrightarrow$$

$$\Leftrightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0 \rightsquigarrow \lambda = 1$$

$$\begin{array}{l|l} -\lambda^3 + 6\lambda^2 - 9\lambda + 4 & \lambda - 1 \\ \hline \lambda^3 - \lambda^2 & -\lambda^2 + 5\lambda - 4 \\ 5\lambda^2 - 9\lambda + 4 & \\ -5\lambda^2 + 5\lambda & \\ -4\lambda + 4 & \\ 4\lambda - 4 & \\ 0 & \end{array}$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (\lambda - 1)(-\lambda^2 + 5\lambda - 4)$$

$$-\lambda^2 + 5\lambda - 4 = 0$$

$$\rightsquigarrow \lambda = \frac{-5 \pm \sqrt{25 - 16}}{-2} = \frac{-5 \pm 3}{-2} \begin{array}{l} \nearrow 4 \\ \searrow 1 \end{array}$$

So $\lambda_1 = 1 = \lambda_2$, $\lambda_3 = 4$.

Eigenvectors:

$\underline{\lambda = 1}$: $M - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

But $\vec{u}_1 \cdot \vec{u}_2 \neq 0$!

→ Gram-Schmidt:

$\vec{v}_1 = \vec{u}_1$

$\vec{v}_2 = \vec{u}_2 - \left(\frac{\vec{v}_1 \cdot \vec{u}_2}{\|\vec{v}_1\|^2}\right)\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$ (check $\vec{v}_1 \cdot \vec{v}_2 = 0$)

So

$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$, $\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{\frac{1}{2} + 1}}\begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$

Finally,

$$M - 4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ \hphantom{xxxxxxx}\uparrow \\ \hphantom{xxxxxxx}\text{free} \end{array}$$

So $x_3 = \alpha$

$x_2 = x_3 = \alpha$

$-2x_1 = -\alpha - \alpha \to x_1 = \alpha$

$$\left\{ \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow \vec{q}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right.$$

So $Q = \left[\vec{q}_1 \,\middle|\, \vec{q}_2 \,\middle|\, \vec{q}_3 \right]$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ //. (Check that $\vec{q}_1 \cdot \vec{q}_3 = 0 = \vec{q}_2 \cdot \vec{q}_3$).

- Why is this theorem also called Principal Axis Theorem?

We can identify symmetric matrices with quadratic forms, that is, (homogeneous) polynomials of the form

$ax^2 + 2bxy + y^2 = 1$ (in 2d)

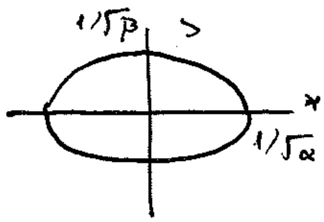
Since $ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

So given a 2x2 matrix, we can think of the corresponding polynomial $\vec{x}^T M \vec{x}$.

<u>Example</u>: Draw the ellipse $5x^2 + 8xy + 5y^2 = 1$.

<u>Sol</u>: This is not easy right now.

We <u>know</u> how to draw ellipses given by $\alpha x^2 + \beta y^2 = 1$



$$\left(\frac{x}{1/\sqrt{\alpha}}\right)^2 + \left(\frac{y}{1/\sqrt{\beta}}\right)^2 = 1$$

But notice that

$$5x^2 + 8xy + 5y^2 = 1 = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}_{M \text{ symmetric.}} \begin{bmatrix} x \\ y \end{bmatrix}$$

So we can write $M = QDQ^T$. In this particular case,

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \to \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

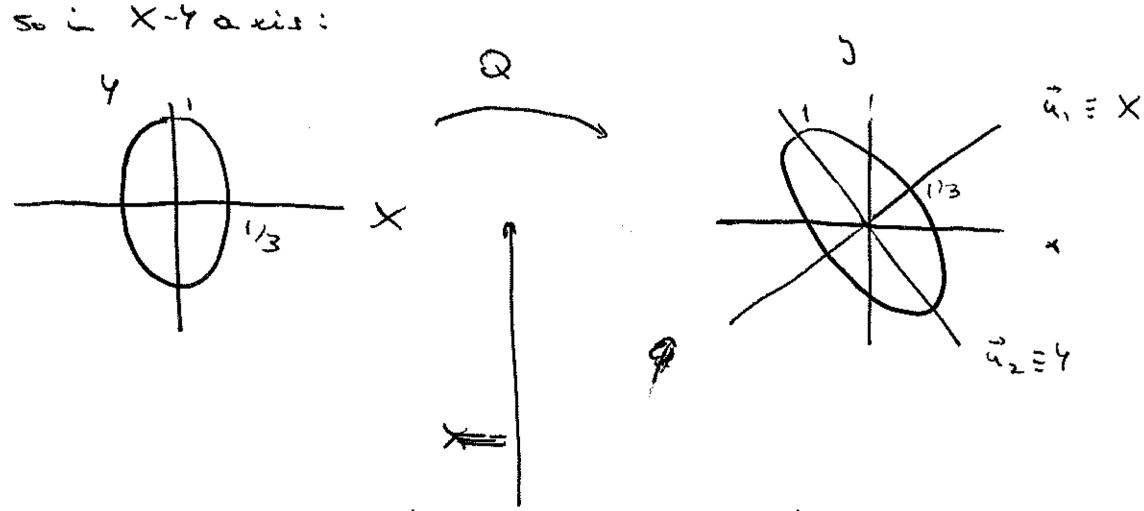
That is, the ellipse is given by

$$\vec{x}^T M \vec{x} = \vec{x}^T Q D Q^T \vec{x} = (Q^T\vec{x})^T D (Q^T\vec{x})$$

If we change variables $\begin{bmatrix} X \\ Y \end{bmatrix} = \vec{X} = Q^T\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow$

$$\rightsquigarrow \left.\begin{aligned} X &= \frac{x+y}{\sqrt{2}} \\ Y &= \frac{x-y}{\sqrt{2}} \end{aligned}\right\} \quad \text{ellipse} \equiv \; 9X^2 + Y^2 = 1,$$

so in X-Y axis:



This is a linear (or orthonormal) transformation.

$$\begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \rightsquigarrow Q\begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{c|c} \vec{u}_1 & \vec{u}_2 \end{array}\right]\begin{bmatrix} 1/3 \\ 0 \end{bmatrix} = \frac{1}{3}\vec{u}_1$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow Q\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \cdots = \vec{u}_2$$

*) Linear transformation given by Q orthogonal:

→ Preserves lengths: $\|Q\vec{x}\|^2 = (Q\vec{x})^T(Q\vec{x}) = \vec{x}^T Q^T Q \vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2$

→ Preserves angles: $(Q\vec{x})\cdot(Q\vec{y}) = \vec{x}^T Q^T Q \vec{y} = \vec{x}^T\vec{y} = \vec{x}\cdot\vec{y} \Rightarrow \cos(\widehat{\vec{x},\vec{y}}) = \cos(\widehat{Q\vec{x},Q\vec{y}})$

VII) Positive semidefinite matrices

This is a preparation to SVD.

<u>Def</u>: Positive semidefinite matrix.

~~A matrix M~~ An $n\times n$ symmetric matrix is called positive semidefinite if

$$\vec{x}^T M \vec{x} \geq 0 \quad \forall \vec{x} \in \mathbb{R}^n .$$

<u>Def</u>: M is called positive definite if $\vec{x}^T M \vec{x} > 0 \quad \forall \vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$.

<u>Proposition</u>: M is posit. semidef. $\Leftrightarrow$ all λ's ≥ 0.

<u>Proof</u>.

1) $\boxed{\Rightarrow}$ Let λ be eig: $M\vec{u} = \lambda \vec{u} \Rightarrow \vec{u}^T M \vec{u} = \lambda \vec{u}^T \vec{u} \Rightarrow$

$$\Rightarrow \lambda \|\vec{u}\|^2 = \vec{u}^T M \vec{u} \geq 0 \Rightarrow \lambda \geq 0 .$$

2) $\boxed{\Leftarrow}$ Let all $\lambda \geq 0$.

M symmetric $\Rightarrow M = QDQ^T$

Any $\vec{x}$ can be written in the basis $\{\vec{q}_1, \dots, \vec{q}_n\}$:

$\vec{x} = \alpha_1 \vec{q}_1 + \dots + \alpha_n \vec{q}_n$. Then,

$$\vec{x}^T M \vec{x} = (\alpha_1 \vec{g}_1^T + \dots + \alpha_n \vec{g}_n^T)(\lambda_1 \vec{g}_1 \vec{g}_1^T + \dots + \lambda_n \vec{g}_n \vec{g}_n^T)(\alpha_1 \vec{g}_1 + \dots + \alpha_n \vec{g}_n) =$$

$$= (\alpha_1 \vec{g}_1^T + \dots + \alpha_n \vec{g}_n^T)(\lambda_1 \alpha_1 \vec{g}_1 + \lambda_2 \alpha_2 \vec{g}_2 + \dots + \lambda_n \alpha_n \vec{g}_n) =$$

$$= \lambda_1 \alpha_1^2 + \dots + \lambda_n \alpha_n^2 \geq 0$$

$$\uparrow$$

$$\lambda_1, \dots, \lambda_n \geq 0 \,//\,.$$

<u>Exercise</u>: ~~The matrices~~ Let A be an $m \times n$ matrix with $r = \mathrm{rank}(A)$. Show that

1) AA^T and A^TA have the same nonzero eigenvalues.

2) $\mathrm{rank}(AA^T) = \mathrm{rank}(A^TA) = r$.

3) Both are positive semidefinite.

<u>Sol</u>:

1) Let $\lambda, \vec{u}$ eig. of A^TA ($\lambda \neq 0$): $A^TA\vec{u} = \lambda\vec{u}$

Then,

$$AA^TA\vec{u} = \lambda A\vec{u} \rightsquigarrow (AA^T)(A\vec{u}) = \lambda(A\vec{u}).$$

So, <u>if</u> $A\vec{u} \neq 0$, then $A\vec{u}$ is eigenvector of AA^T with eigenvalue <u>λ</u>.

But if $A\vec{u} = 0$, then $\lambda = 0$ (and we said $\lambda \neq 0$).

We can do the same starting from AA^T //.

2) $\text{rank}(AA^T) = \text{rank}(A^TA)$ from 1) and 3).

Now, let's show $\text{rank}(A^TA) = r = \text{rank}(A)$.

By rank-nullity theorem, since A^TA is $n \times n$ as A, we just need to show that $\dim(N(A^TA)) = \dim N(A)$.

But $N(A^TA) = N(A)$!

$\hookrightarrow \vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow A^TA\vec{x} = \vec{0} \Rightarrow \vec{x} \in N(A^TA)$.

$\hookrightarrow \vec{x} \in N(A^TA) \Rightarrow A^TA\vec{x} = \vec{0} \Rightarrow A\vec{x} \in N(A^T) = C(A)^\perp$.

But $A\vec{x} \in C(A) \longrightarrow$ So $A\vec{x} = \vec{0} \Rightarrow \vec{x} \in N(A)$ //.

3) $\vec{x}^TA^TA\vec{x} = (A\vec{x})^TA\vec{x} = \|A\vec{x}\| \geq 0 \quad \forall \vec{x}$

$\vec{x}^TAA^T\vec{x} = (A^T\vec{x})^TA^T\vec{x} = \|A^T\vec{x}\| \geq 0 \quad \forall \vec{x}$.

$\boxed{\text{Chapter 7}}$: Singular Value Decomposition.

We will see that this decomposition has plenty of applications.
All of them share one idea: the information contained on a big matrix is usually reduced, as many columns are "almost" linearly dependent.

Think of this matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \operatorname{rank}(A) = 1 .$$

and the rows are "almost" lin. dependent

$$B = \begin{bmatrix} 1.1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1.1 & 1.2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0.2 & 0 & 0 \end{bmatrix}}_{\text{are "negligible"}}$$

B has $\operatorname{rank}(B) = 3$, but we would like a method that tells us that $\operatorname{rank}(B) \approx 1$ (that is, that detected the "main" part of B).

So, our goal will be to find how to decompose a matrix as a sum of rank one matrices, in a way that we can see which are the important pieces.

<u>Goal</u>: Given A $m \times n$, find

$$A = U \Sigma V^T = \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T,$$

where U is orthogonal $m \times m$ matrix.

V is orthogonal $n \times n$ matrix.

Σ is $m \times n$ diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$.

$$U: \begin{array}{cc} & m \\ m & \left[\begin{array}{c|c|c} \vec{u}_1 & \cdots & \vec{u}_m \end{array}\right] \end{array} \quad \Sigma: \begin{array}{cc} & r \\ m & \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ \hline & & & \end{array}\right] \end{array}, \quad V: \begin{array}{cc} & n \\ n & \left[\begin{array}{c|c|c} \vec{v}_1 & \cdots & \vec{v}_n \end{array}\right] \end{array}$$

I'll first say how to find U, Σ, V and then we will show why:

1) $\underline{\underline{\Sigma}}$

First, find the nonzero eigenvalues of A^TA and AA^T.

↳ They are the same, so use the smaller of those matrices!

Order them $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$

↑

r nonzero eigenvalues (positive since AA^T is semidef.).

Then,

let $\sigma_i = \sqrt{\lambda_i}$ and
$$\Sigma = \begin{array}{c@{}c} \begin{array}{cc} r & n-r \end{array} & \\ \left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ \hline & & & & 0 \ddots \end{array}\right] & \begin{array}{c} r \\ \\ \\ \\ m-r \end{array} \end{array}$$

<u>Example:</u>

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \sim \lambda_1 = 3,\ \lambda_2 = 1 \rightarrow \begin{array}{l} \sigma_1 = \sqrt{3} \\ \sigma_2 = 1 \end{array}.$$

$$\text{So } \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

2) $\underline{\underline{V}}$

Find orthonormal eigenvectors for A^TA :

$\rightarrow r$ from $\lambda_1, \ldots, \lambda_r \rightsquigarrow \vec{v}_1, \ldots, \vec{v}_r$

$\rightarrow n-r$ from $\lambda = 0 \rightsquigarrow \vec{v}_{r+1}, \ldots, \vec{v}_n$

(• Don't forget to make them unit size
• You might need Gram-Schmidt.)

$$V = \left[\begin{array}{ccc|ccc} \vec{v}_1 & \cdots & \vec{v}_r & \vec{v}_{r+1} & \cdots & \vec{v}_n \end{array}\right] \text{ (orthogonal)}.$$
$n \times n$

3) $\underline{\underline{U}}$

For $i = 1, \ldots, r$, let $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$.

Then, find $m-r$ ~~eig~~ orthonormal eigenvectors corresponding to $\lambda = 0$. That is, find an orthonormal basis for $N(AA^T)$ $(= N(A^T))$.

$$U = \left[\begin{array}{c|c|c|c|c|c} \frac{A\vec{v}_1}{\sigma_1} & \cdots & \frac{A\vec{v}_r}{\sigma_r} & \vec{u}_{r+1} & \cdots & \vec{u}_n \end{array}\right] \text{ (orthogonal)}$$
$m \times m$

$(A = U\Sigma V^T \Rightarrow AV = \Sigma U)$

To complete the previous example, U, V?

- $V \to$ eigenvectors of $A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $(\lambda_1 = 3, \lambda_2 = 1)$

$A^TA - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (unit size).

$A^TA - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sim \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (unit size)

So $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

- U: it is a 3×3 matrix, and $r = \text{rank}(A) = 2 \to$

$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \to$ (this are already unit size; we'll see why).

$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Finally, $\vec{u}_3$ is an eigenvector of AA^T corresponding to $\lambda = 0$:

$AA^T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \underset{\uparrow \text{ free}}{0} \end{bmatrix} \to \begin{matrix} x_3 = \alpha \\ x_2 = -\alpha \\ x_1 = \alpha \end{matrix}$

so $\vec{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

In summary, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$,

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

¶ One can check $A = U\Sigma V^T$; columns of U orthonormal; columns of V orthonormal //