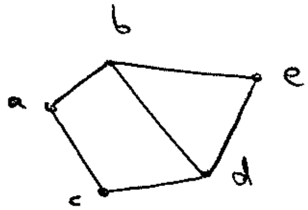
MATH 312 LECTURE 14 : Eigenvalues and eigenvectors.

I] Introduction.

Let's try to solve the following problem. Given the graph



how many paths connecting a to e?

We can define the graph using the adjacency matrix:

$$A = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}$$

There is a 1 only if you can get from one node to the other with a path of length one.

Look at A^2:

$$A^2 = \begin{bmatrix} 2 & 0 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Each entry "coincides" with the number of paths of length 2 connecting the two nodes.

One can check that the entry i,j of A^k is the n° of paths of length k joining nodes i and j.

So, if we know A^k for all k we know how to compute the number of paths... But how to compute A^{1000}, for example, if the graph represents something bigger, like cities in US?

→ The main purpose of eigenvalues/eigenvectors (that arises in most applications) is to compute powers of A in an efficient way.

- Let's start with diagonal matrices:

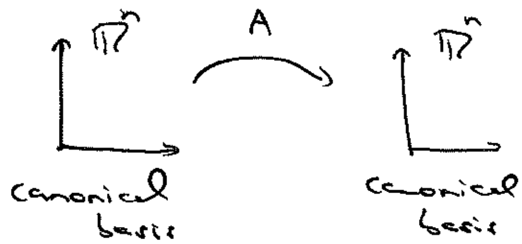
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \to A^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \to \dots \to$$

$$\to A^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \quad \underline{\underline{\text{easy}}}. \quad (\text{same for } A \ n\times n \text{ matrix})$$

But, recall that an $n \times n$ matrix represent a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ (with chosen basis, for example, the canonical one).

<u>Question</u>: Can we find a new basis for $\mathbb{R}^n$ in which a given general matrix A becomes diagonal?

That is, if

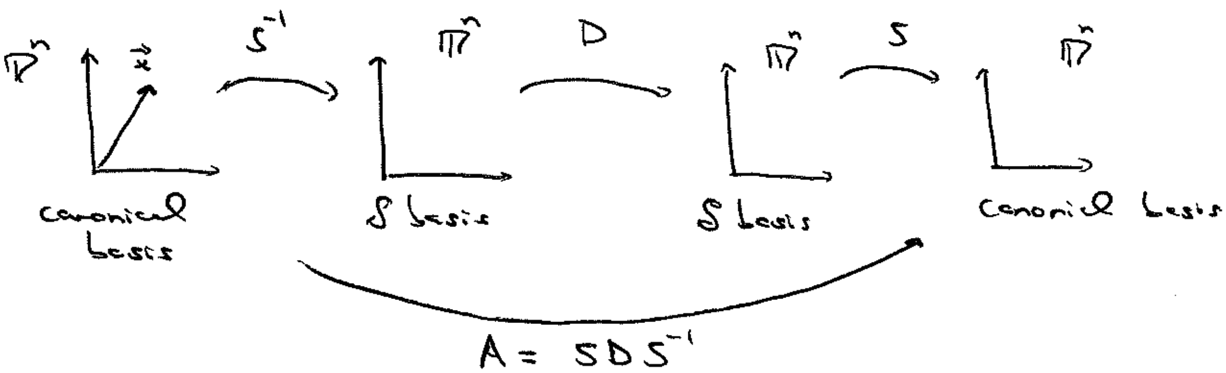


A non diagonal,

can we find a basis $\mathcal{S} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ such that the linear application above is represented by a diagonal basis?

- If the answer is yes, we would have an invertible matrix

$$S = \left[\vec{u}_1 \middle| \cdots \middle| \vec{u}_n\right] \quad \text{such that} \quad A = SDS^{-1}$$



Note that $S = \left[\vec{u}_1 \middle| \dots \middle| \vec{u}_n\right]$ is the matrix that changes coordinates of a vector written in the S basis to the canonical one:

$$\vec{x} = c_1\vec{u}_1 + \dots + c_n\vec{u}_n = \underbrace{\left[\vec{u}_1 \middle| \dots \middle| \vec{u}_n\right]}_{S} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{x}\Big|_{\substack{\text{canonical} \\ \text{basis}}}$$

vectors written in the canonical basis.

$\vec{x}\big|_S \equiv$ coordinates of $\vec{x}$ in the basis S

- Going back, we wanted to compute A^k. If we assume that we can obtain $A = SDS^{-1}$, then

$$A^k = SD\underbrace{S^{-1}S}D\underbrace{S^{-1}S}DS^{-1} \dots SDS^{-1} = SDDD\dots DS^{-1} = SD^kS^{-1} \quad //.$$

diagonal, so easy!

- OK, so how do we find such matrices S, D?

Let's see what characterizes diagonal matrices.

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ Notice what D does to the basis vectors:

$$\left.\begin{aligned} D\vec{e}_1 &= \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\lambda_1\\ 0\end{bmatrix} = \lambda_1\vec{e}_1 \\ D\vec{e}_2 &= \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ \lambda_2\end{bmatrix} = \lambda_2\vec{e}_2 \end{aligned}\right\}$$

That is, we want to find those vectors $\vec{u}$ that don't change their direction when multiplied by A:

$$\| \; A\vec{u} = \lambda\vec{u} \; \|$$

Those $\vec{u}$ will give the basis on which the linear application given by A (in the canonical basis) becomes diagonal.

II) Eigenvalues and eigenvectors

- *Def*: We say that $\vec{u} \neq \vec{0}$ is an eigenvector of A with corresponding eigenvalue λ if

  $A\vec{u} = \lambda\vec{u}$.

- How to find λ and $\vec{u}$?

$A\vec{u} = \lambda\vec{u}$ is nonlinear, since both λ and $\vec{u}$ are unknowns.

But

$A\vec{u} = \lambda \vec{u} \iff A\vec{u} - \lambda\vec{u} = \vec{0} \iff (A - \lambda I)\vec{u} = \vec{0}$. That is,

$\vec{u} \in N(A-\lambda I)$ and, $\vec{u} \neq \vec{0}$ (by definition), so

$\| A - \lambda I$ has to be singular $\|$ (its nullspace is not the trivial $\{\vec{0}\}$).

Therefore,

$\| \underbrace{\det(A-\lambda I)}_{\text{Characteristic polynomial of } A \text{ (degree } n)} = 0 \|$ → This gives us an equation on λ (of order n)

↓ Solve it to find

n eigenvalues (→ Can be complex! → Can be repeated!)

Example:

1) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

So $\det(A-\lambda I) = (1-\lambda)(4-\lambda) - 4 = 4 + \lambda^2 - 5\lambda - 4 = \lambda(\lambda - 5)$,

$\det(A - \lambda I) = 0 \iff \lambda(\lambda-5) = 0 \iff \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 5 \end{cases}$.

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \rightsquigarrow$

$$\det(A - \lambda I) = 0 \iff \lambda^2 + 1 = 0 \iff \lambda^2 = -1 \iff \begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$$

Remark: It is not a surprise that this matrix doesn't have real eigenvalues. Why?

$A\vec{u} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \rightsquigarrow$ it is a rotation of $\vec{u}$ by 90°

That is, any vector $\vec{u}$ gets rotated by A!

But an eigenvector can only be scaled: $A\vec{u} = \lambda\vec{u}$!!

- So, we first find the λ's by solving an equation of degree n. Then we find the eigenvectors:

$(A - \lambda_1 I)\vec{u}_1 = \vec{0} \rightsquigarrow \vec{u}_1 \in N(A - \lambda_1 I)$ we already know how to find the $\vec{u}$'s !

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\lambda_1 = 0$, $\lambda_2 = 5$. Eigenvectors?

$$A - \lambda_1 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow (A - \lambda_1 I)\vec{u} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(any multiple is valid)

$$A - \lambda_2 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Exercise: If A has eigenvalues λ_1, λ_2, what are the eigenvalues of A^2? and of $A + I$? of A^{-1}?

Sol: Let $\vec{u}_1, \vec{u}_2$ be the corresponding eigenvectors of A:

$$A\vec{u}_1 = \lambda_1 \vec{u}_1$$

Then,

- $AA\vec{u}_1 = \lambda_1 A\vec{u}_1 = \lambda_1^2 \vec{u}_1 \Rightarrow \lambda_1^2$ is an eigenvalue of A^2 (with same eigenvector $\vec{u}_1$)

- $(A + I)\vec{u}_1 = A\vec{u}_1 + \vec{u}_1 = \lambda_1\vec{u}_1 + \vec{u}_1 = \underbrace{(\lambda_1 + 1)}_{\text{eigenvalue of } A+I}\vec{u}_1$.

- $A^{-1}A\vec{u}_1 = \lambda_1 A^{-1}\vec{u}_1 \Rightarrow A^{-1}\vec{u}_1 = \frac{1}{\lambda_1}\vec{u}_1$

  ↳ eigenvalue of A^{-1}.

• <u>Remark</u>: If A has an eigenvalue equal to zero,
then A does not have an inverse.

(↳ $A\vec{u} = \vec{0}$ for $\vec{u} \neq \vec{0}$)

the corresponding eigenvector is in $N(A)$.

• <u>Remark</u>: Eigenvectors corresponding to nonzero eigenvalues are trivially in $C(A)$.

$A\vec{u} = \underbrace{\lambda\vec{u}}$

this is "output" by A (it's a linear comb. of its columns.).

• <u>Exercise</u>: If A has an eigenvalue λ and B an eig. β,
is $\lambda\beta$ an eigenvalue of AB?

False in general.

$$\left(\left.\begin{array}{l} A\vec{u}_1 = \alpha\vec{u}_1 \\ B\vec{u}_2 = \beta\vec{u}_2 \end{array}\right\} \to AB\vec{u}_2 = A\beta\vec{u}_2 \; ! \right)$$

this is not, in general, an eigenvector of A.

Important and useful:

$\det(A) =$ product of eigenvalues.

$\text{trace}(A) =$ sum of eigenvalues (trace $(A) =$ sum of diagonal entries).

III) Diagonalization.

• Theorem: If A has n l.i. eigenvectors $\vec{u}_1, \dots, \vec{u}_n$, then

$$A = SDS^{-1} \quad \text{where} \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ (eigenvalues)}$$

$$S = \begin{bmatrix} \vec{u}_1 & | \cdots | & \vec{u}_n \end{bmatrix} \text{ (eigenvectors)}$$

Proof: $\vec{u}_1, \dots, \vec{u}_n$ l.i. $\Rightarrow \exists S^{-1}$.

So just need to check $AS = SD$:

$$AS = A\underbrace{\begin{bmatrix} \vec{u}_1 & | \cdots | & \vec{u}_n \end{bmatrix}}_{S} = \begin{bmatrix} \lambda_1 \vec{u}_1 & | \cdots | & \lambda_n \vec{u}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{u}_1 & | \cdots | & \vec{u}_n \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \quad //.$$

Examples: Find diagonalization of A :

1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Eigenvalue: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$ $\nearrow \lambda_1 = 1$, $\searrow \lambda_2 = 2$.

Remark: For triangular matrices, the eigenvalues are the diagonal elements.

(of course! recall det of triangular matrix)

Eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(same order!)

2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenvalues → $\lambda_1 = 1 = \lambda_2$ (repeated)

Eigenvectors: $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

rank$(A-I) = 1$, so its nullspace has dimension 1. →

→ Only 1 eigenvector! It is not possible to diagonalise A.

• <u>Remark</u>: Possible cases (Summary)

1) All eigenvalues are distinct: Then all eigenvectors are l.i., therefore we can diagonalize. [(*) proof in -143-] (optional)

2) Some repeated eigenvalues: We have to compute the nullspace of those repeated eigenvalues.

↳ If we obtain full set of eigenvectors → diagonalise ok ✓.

↳ If we don't → diagonalisation not possible ✗.

(*)(proof) Distinct eigenvalues $\rightarrow$ l.i. eigenvectors :

Assume it is false, i.e.,

$\vec{u}_1, \vec{u}_2$ are two eigenvectors corresponding to $\lambda_1 \neq \lambda_2$ and they are l.d. Then,

[1] $c_1 \vec{u}_1 + c_2 \vec{u}_2 = \vec{0}$ for some c_1, c_2 <u>not</u> both zero.

But, multiply [1] by A :

$$c_1 A \vec{u}_1 + c_2 A \vec{u}_2 = 0 \rightarrow c_1 \lambda_1 \vec{u}_1 + c_2 \lambda_2 \vec{u}_2 = \vec{0}$$

Multiply [1] by λ_2 :

$$c_1 \lambda_2 \vec{u}_1 + c_2 \lambda_2 \vec{u}_2 = \vec{0} \longrightarrow c_1(\lambda_1 - \lambda_2)\vec{u}_1 = 0 \rightarrow \underline{\underline{c_1 = 0}}$$

$\lambda_1 \neq \lambda_2$

$\vec{u}_1 \neq \vec{0}$ (by def. of eigenvector)

Multiply [1] by λ_1 :

$$c_1 \lambda_1 \vec{u}_1 + c_2 \lambda_1 \vec{u}_2 = \vec{0} \longrightarrow c_2(\lambda_2 - \lambda_1)\vec{u}_2 = \vec{0} \rightarrow \underline{c_2 = 0}$$

So we conclude $c_1 = c_2 = 0$! Contradiction //.

(the proof can be extended to more eigenvalues using the same idea...)

• Exercise: $A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ find diagonalization.

Hint: rank $(A) = 1$.

Sol:

rank $(A) = 1 \Rightarrow \dim N(A) = 2 \Rightarrow \lambda_1 = \lambda_2 = 0$

Now,

trace $(A) = 6 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3 = 6$.

Eigenvectors:

$$A - \lambda_1 I = A \sim \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} \sim \quad \vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} -4 & 1 & 2 \\ 4 & -4 & 4 \\ 2 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 3/2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & \underset{\uparrow \text{ free}}{0} \end{bmatrix}$$

$$\rightarrow \vec{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

so

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^{-1} //.$$