MATH 312   LECTURE 13

: Determinants ($\sim$ Chap. 5)

This lecture is intended as a reminder of known things.

I| Properties ($\sim$ Sec. 5.1)

• Def: Determinant

The determinant is the only function from the vector space of $\underline{\underline{n \times n}}$ matrices to real numbers,

$$\left.\begin{array}{rcl} \det : n\times n \text{ matrices} & \longrightarrow & \mathbb{R} \\ A & \longmapsto & \det(A) \equiv |A| \end{array}\right\}$$

that satisfies the following three properties:

1) Normalization: $\det(I) = 1$ (I identity matrix of size $n \times n$)

2) Antisymmetry: Exchange of two rows changes the sign.

Example: $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3) Multilinearity: The determinant is a linear function of each row separately (i.e., keeping the rest constant).

That is, multilinearity means that:

3.1) Scaling a row scales the determinant,

$$\begin{vmatrix} \lambda a & \lambda b \\ c & d \end{vmatrix} = \lambda \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

<u>Remark</u>: Note that this implies that

$$\begin{vmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{vmatrix} = \lambda^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ and in general,

$$\| \det(\lambda A) = \lambda^n \det(A) \|$$ (important and source of typical mistakes).

3.2) Addition in one row is linear:

$$\begin{vmatrix} a+\tilde{a} & b+\tilde{b} \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \tilde{a} & \tilde{b} \\ c & d \end{vmatrix}$$

. <u>Remark</u>: Note that we are <u>defining</u> the determinant by its properties. You can "check" the last one (for example) using the "formula"

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

but this formula is <u>derived</u> from the ~~form~~ properties.

• <u>Remark</u>: The previous "definition" is indeed a definition because it can be proved that such a function is unique (not too difficult to prove, not too interesting either: google).
(or office hours...)

Right now we "don't know" how to compute determinants yet for a particular matrix.
Let's derive some useful properties (both for theoretical and computing reasons):

4) If two rows are equal, then the determinant is zero.
Let $\tilde{A}$ be the matrix A with the two equal rows exchanged. Then
$\det \tilde{A} = -\det A$. But clearly $\tilde{A} = A$, so
$\det A = -\det A \Rightarrow 2\det A = 0 \Rightarrow \det A = 0$.

5) Elimination steps don't change the det:

$$\begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix} \underset{\text{prop. 3.2)}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -\ell a & -\ell b \end{vmatrix} \underset{\text{prop. 4)}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \ell \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0}$$

6) If A has a row of zeros, the $\det A = 0$.

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \underset{\text{prop. 5)}}{=} \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

(same proofs for $n \times n$ matrices).

7) Prop. 5) and 6) $\Rightarrow$ If a matrix has some linearly dependent rows, then its det is zero.

8) For triangular matrices, the determinant is equal to the product of its diagonal entries.

If the diagonal entries are not zero, then by elimination one obtains

$$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} \underset{\text{prop. 5)}}{=} \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} \underset{\text{prop. 3.1)}}{=} a_{11} \begin{vmatrix} 1 & 0 \\ 0 & a_{22} \end{vmatrix} \underset{\text{prop. 3.1)}}{=} a_{11} a_{22} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \underset{\text{prop. 1)}}{=} a_{11} a_{22}$$

If a diagonal entry is zero, elimination on the rest of columns will produce a row of zeros:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \right) \underset{\text{prop. 5)}}{=} \begin{vmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \underset{\text{prop. 6)}}{=} 0$$

9) $\det(A) = \pm$ product of pivots (we allow pivots to be zero here).

↳ By elimination steps and some possible row exchanges (which give the $\pm$) we go from A to U (upper triangular with diagonal entries = pivots).

<u>Example</u>:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

<u>Remark</u>: This is the way computers calculate $\det(A)$.
It is the most efficient way for big matrices.

10) A is invertible $\Leftrightarrow \det(A) \neq 0$

[Recall that A was invertible if we had n pivots ($\neq 0$)]

This the most important property for us.

11) $\det(AB) = \det(A)\det(B)$ [Very useful]

(Proof: in the book if interested).

12) $\det(A^T) = \det(A)$

$\hookrightarrow$ All the previous properties can now be stated in terms of columns instead of rows.

(i.e., if a column is all zeros, the determinant is zero; if some column are lin. dep., the det is zero; ...).

13) $\det(A) \neq 0 \Leftrightarrow$ columns of A are linearly independent.
(rows)

(We can also see this through pivots)

14) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

This is a consequence of 11):

$$\left.\begin{aligned} \det(AA^{-1}) &= \det(I) = 1 \\ &= \det(A)\det(A^{-1}) \end{aligned}\right\} \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

• Example:

1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det(A)$?

Sol: Assume $a \neq 0$. Then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{vmatrix} = ad - cb.$$

If $a = 0$,

$$\begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ 0 & b \end{vmatrix} = -cb.$$

So, in any case, $\| \det(A) = ad - cb \|$.

2) We know that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{when } \det(A) \neq 0).$$

Check that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Sol:

$$\det(A^{-1}) = \det\left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{(ad-bc)^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} =$$

$$= \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} \,//.$$

• <u>Exercise</u>: Find the error:

Let $CD = -DC$. Then, $|C||D| = -|D||C|$, thus

$2|C||D| = 0$ and therefore $|C| = 0$ or $|D| = 0$.

Sol: Error is that $CD = -DC \not\Rightarrow |C||D| = -|D||C|$ in general.

Recall that

$\det(\lambda A) = \lambda^n \det(A)$, so $\det(-A) = (-1)^n \det(A)$.

In this case, we would obtain

$\det(CD) = \det(-DC) \Rightarrow |C||D| = (-1)^n |D||C|$.

So if n is even, $|C|$ and $|D|$ can be different than zero.

• <u>Exercise</u>: Find the error

$$P = A(A^TA)^{-1}A^T \Rightarrow |P| = |A| \frac{1}{|A^T||A|} |A^T| = 1 .$$

<u>Sol</u>: The matrix A is not square in most cases, so it is meaningless to take determinants.

• <u>Exercise</u>: True / False

1) If A is not invertible, then AB is not invertible.

True: $\det(AB) = \det(A)\det(B) = 0 \Rightarrow AB$ not invertible.

A not invertible
$\Leftrightarrow \det(A) = 0$

2) $\det(A)$ is always equal to the product of its pivots.

False: $\det(A) = \pm$ product pivots.

3) $\det(A+B) = \det(A) + \det(B)$

False: $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4 \neq \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

Determinant as volume

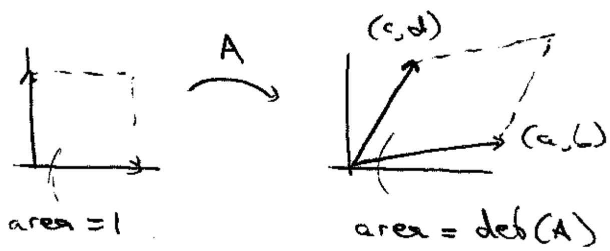
One can see that the volume (n-dimensional) satisfies the same properties of the determinant, so, since that function was unique, they are the same thing.

That is, if

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ represent a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (with canonical basis), then

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

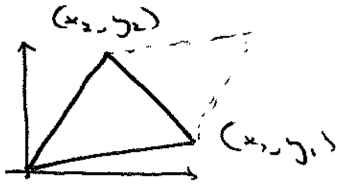
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$



(Note: see google for some nice visualisations in 3D).

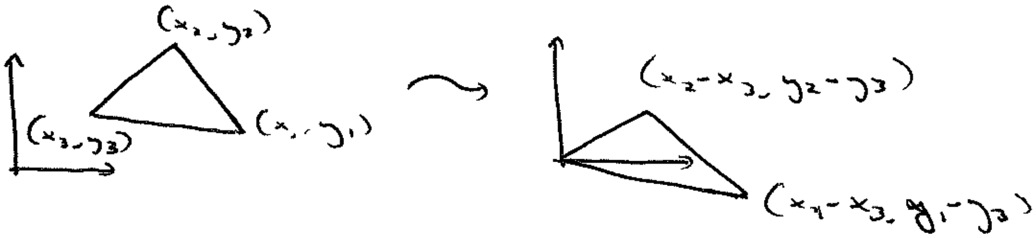
__Remark__: Notice that if we scale a column by λ, the det. gets multiplied by λ. This corresponds to multiplying one ~~the~~ side of the square by λ, the volume (area in 2D) gets multiplied by λ.

We can use this to find the formula for the area of a general triangle knowing the vertices:



$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

Similarly, for

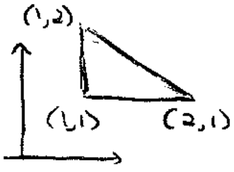


$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} =$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_3 & x_3 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} =$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_3 & y_3 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_3 & x_3 \\ y_1 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & \cancel{x_3} \\ \cancel{y_3} & y_3 \end{vmatrix}$$

(This is just to check that it coincides with the formula on the book. For a real example $x_1, \dots y_3$ are all numbers! So we just compute the 2x2 determinant (instead of the 3x3 in the book):



Area $= \frac{1}{2}$. Using the formula: $\frac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$ ✓)

II] Cofactors (~ Sec 5.2)

Let A be an $n \times n$ matrix. We called cofactor (i,j) of A to the following number:

$$\boxed{C_{ij} = (-1)^{j+i} \det M_{ij}}$$, where $M_{ij} = A$ without row i and column j

Using cofactors, the determinant is given by:

$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$, and also ← expansion through row i.

$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$ ← expansion through column j

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

We can calculate $\det(A)$ by cofactors through any row or column:

1) $\underbrace{\det A}_{\text{expansion using row 1}} = 1 \cdot \begin{vmatrix} 3 & 4 & 5 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 & 4 \\ 5 & 4 & 0 \\ 2 & 0 & 0 \end{vmatrix} = \underbrace{1 \cdot 1 \begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix}}_{\text{expansion through row 3}} - \underbrace{2 \cdot 2 \begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix}}_{\text{expansion through row 3}} = 48$

<u>Remark</u>: This is only useful when a row or column has a lot of zeros.

In the example above we could have used column 3:

$$\det A = (-1)^{2+3} \cdot 4 \begin{vmatrix} 1 & 0 & 2 \\ 5 & 4 & 3 \\ 2 & 0 & 1 \end{vmatrix} = -4 \cdot 4 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -4 \cdot 4 \cdot (-3) = 48 .$$

expansion by column 2.

- Let A be 3×3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

By cofactors we obtain the well-known formula

$$\det(A) = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} +$$

$$- a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

To remember it

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{32} & a_{33} \end{vmatrix}$$

But why is that formula (and the cofactors one) true?

Let's see it in the 3×3 case:

Think of the first row as

$$[a_{11} \quad a_{12} \quad a_{13}] = [a_{11} \quad 0 \quad 0] + [0 \quad a_{12} \quad 0] + [0 \quad 0 \quad a_{13}]$$

so that by prop. 3.1)

$$|A| = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Now we repeat it with the second row (we would obtain 3×3 determinants!)

lin. dep.

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}_{=0} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

From each one, three more through 3^{rd} row: $3\times3\times3 = 27$!!.

But most of them are zero!

Indeed,

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{vmatrix}}_{=0} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

If we do the same with the rest, we notice that the only ones different from zero are those which have one and only one element in each row and column:

$$\det(A) = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + \cdots$$

Example:

$$\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & & & \\ & & 4 & \\ & 4 & & \\ & & & 1 \end{vmatrix} + \begin{vmatrix} & & & 2 \\ & & 4 & \\ & 4 & & \\ 2 & & & \end{vmatrix} =$$

$$= - \begin{vmatrix} 1 & & \\ & 4 & \\ & & 4 & \\ & & & 1 \end{vmatrix} + \begin{vmatrix} 2 & & & \\ & 4 & & \\ & & 4 & \\ & & & 2 \end{vmatrix} = 48 .$$

Once we pick 1, we have to choose an element of the second column that is not in the first row (that is, 3, 4 or 0). If we choose 0 that doesn't add anything. If we choose 3, then we have to choose an element of the third column that is not on the first or second row (that is, 0 or 0!) ...

Example:

$$A = \begin{bmatrix} 0 & x & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \qquad \frac{d}{dx}|A| \,?$$

$$|A| = -x \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \rightarrow \frac{d}{dx}|A| = - \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = -3 .$$

cofactors in 1st row

III / Cramer's Rule (~ Sec. 5.3)

Let A be such that $\det(A) \neq 0$. Then $A\vec{x} = \vec{b}$ is solved by

$$x_1 = \frac{\det B_1}{\det A}, \ldots, x_n = \frac{\det B_n}{\det A}, \text{ where } B_j = A \text{ with column } j^{th} \text{ replaced by } \vec{b}$$

• <u>Example</u>: Solve $\left.\begin{aligned} 3x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 2 \end{aligned}\right\}$

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{b}} \Rightarrow \quad x_1 = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{-3}{-1} = -3$$

$$x_2 = \frac{\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{4}{-1} = -4 \, //.$$

→ Why is this rule true?

Write
$$\begin{bmatrix} A \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}}_{\text{has } \det = x_1} = \underbrace{\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}_{B_1} \Rightarrow$$

$$\Rightarrow \det(A)\, x_1 = \det B_1 \Rightarrow x_1 = \frac{\det B_1}{\det A} \quad (\text{if } \det(A) \neq 0) \, //.$$

Exercise: If all entries of A are integers and $\det A = \pm 1$, prove that A^{-1} also has integer entries.

Sol:

Let's find A^{-1} using Cramer's rule: (for a 3×3)

$$A^{-1} = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \rightsquigarrow A\begin{bmatrix} x_1 \\ x_4 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \quad x_1 = \frac{\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}}{\det A} = \frac{C_{11}}{|A|} \text{ (integer) } = (A^{-1})_{1,1}$$

$$x_5 = \frac{\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}}{|A|} = \frac{C_{22}}{|A|} = (A^{-1})_{2,2} \text{ (integer)}$$

$$\vdots$$

//.

Remark: This is the "well-known" (and kind of useless...) formula

$$A^{-1} = \frac{C^T}{\det(A)} \quad \equiv \quad (A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$