MATH 312 LECTURE 23	: Linear Programming

Example: You have 1\$, and you have to invest it on assets A and B, satisfying:

1) You invest in A at least ~~twice~~ double than in B.
2) At least 1/2 \$ is invested.

We can write this constraints as inequalities:

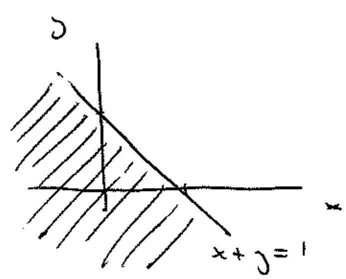
Denote $\left.\begin{array}{l} x = \text{invested on } A \\ y = \text{ " on } B \end{array}\right\}$, then

$$\left.\begin{array}{ll} x + y \leq 1 & (\text{you only have } 1\$) \\ x \geq 2y & (\text{at least double in A than B}) \\ x + y \geq \frac{1}{2} & (\text{at least } 1/2\,\$ \text{ invested}) \end{array}\right\} \text{ and } \left\{\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right. .$$

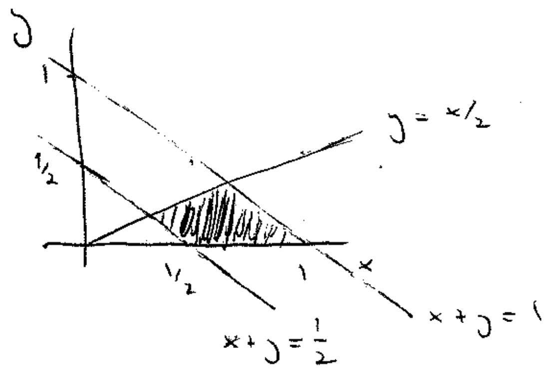
Q: What values are x, y allowed to take?

Note that each inequality represent a half plane:

$x + y \leq 1 \rightarrow$



Drawing all the constraints gives:



This is the <u>feasible</u> region.

We usually denote all the constraints in matrix form:

$A\vec{x} \le \vec{b}$.

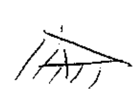
Q: What for $x+y \ge \frac{1}{2}$? $\rightarrow$ $-x-y \le -\frac{1}{2}$.

So,
$$\left.\begin{array}{l} x+y \le 1 \\ -x+2y \le 0 \\ -x-y \le -\frac{1}{2} \end{array}\right\} \rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} \quad (\text{and } x \ge 0, y \ge 0).$$

Def: Feasible region

Intersection of the half-spaces arising from the inequalities.

<u>Remark</u>: It can be · bounded : $x+y \le 1, x \ge 0, y \ge 0$ 

· Unbounded: $x+y \le 1$ 

· empty $x+y \ge 1, x+y \le 0$.

Def. A Linear Programming problem asks to maximize a linear profit $\vec{c}^T\vec{x}$ subject to linear constraints $A\vec{x} \le \vec{b}$ and $\vec{x} \ge 0$, where $A, \vec{c}, \vec{b}$ are data.

- In our example: Say that A has a twofold return while threefold in B.

Then, for an investment of x in A, y in B, we get back

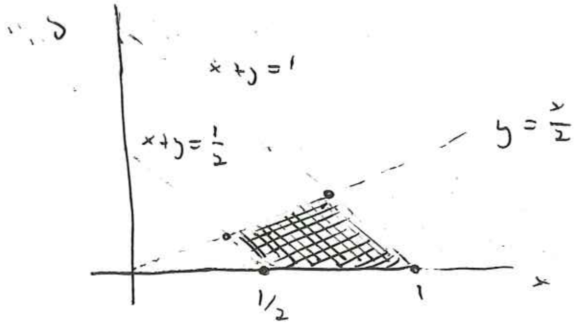
$$2x + 3y = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \to \vec{c}^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

- So the lin. programming problem would be:

$$\underbrace{\left.\begin{array}{ll} \max. & \vec{c}^T\vec{x} \\ \text{s.t.} & A\vec{x} \le \vec{b} \\ & \vec{x} \ge 0 \end{array}\right\}}_{\text{Standard form}} \text{ where } \quad \vec{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}, \; \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ unknowns.}$$

- How to solve it? → Geometric idea (we'll follow our example).

1) Draw the feasible region:

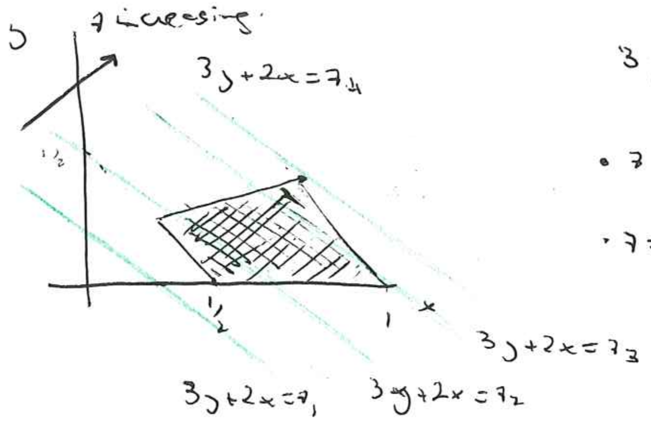


All possible solution are in that region.

2) Maximize the profit: $\vec{c}^T \vec{x} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x + 3y$

↳ Pick a profit z ⇝ then $z = 2x+3y$ describes a line;

each combination of x, y on that line gives same profit.



$3y + 2x = z \rightarrow y = \frac{z}{3} - \frac{2}{3}x$

• $z = 0 \rightsquigarrow y = \frac{-2}{3}x$

• $z = 1 \rightsquigarrow y = \frac{1}{3} - \frac{2}{3}x$

⋮

We "move" the line that gives the profit until we leave the feasible region. We can see that the maximum will always be obtained at a corner (*).

(*) Assuming the feasible region is bounded and that there is a unique solution.

3) Evaluate the profit function $\vec{c}^T\vec{x}$ at all corners and pick the maximum.

In our example,

$(x=\frac{1}{2}, y=0)$, $\qquad \left.\begin{array}{l} x+y=\frac{1}{2} \\ 2y-x=0 \end{array}\right\} \Rightarrow \left(y=\frac{1}{6}, x=\frac{1}{3}\right)$

$(x=1, y=0)$,

$\left.\begin{array}{l} x+y=1 \\ 2y-x=0 \end{array}\right\} \rightarrow \left(y=\frac{1}{3}, x=\frac{2}{3}\right)$

$\hookrightarrow \left(\frac{1}{2}, 0\right) \rightarrow 2\frac{1}{2}+3\cdot 0=1$

$(1,0) \rightarrow 2$

$\left(\frac{1}{3}, \frac{1}{6}\right) \rightarrow 2\cdot\frac{1}{3}+3\frac{1}{6}=\frac{7}{6}$

$\left(\frac{2}{3}, \frac{1}{3}\right) \rightarrow 2\cdot\frac{2}{3}+3\frac{1}{3}=\frac{4}{3}+1=\frac{7}{3}$

$\Rightarrow$ maximum is $\frac{7}{3}$, at point $x=\frac{2}{3}$, $y=\frac{1}{3}$.

(We "saw" it on the picture).

<u>Remark</u>: In higher dimensions, finding all the corners and evaluating the profit is <u>not</u> an efficient method. But the moral is true: the optimum is reached at a corner.

Dual problem

For every linear programming problem, there is a closely related linear programming problem, called its dual.

If the "primal problem" is given in standard form:

$$\left.\begin{array}{ll} \max & \vec{c}^T\vec{x} \\ \text{s.t.} & A\vec{x} \le \vec{b} \\ & \vec{x} \ge \vec{0} \end{array}\right\} \text{ its dual is } \left.\begin{array}{ll} \min & \vec{b}^T\vec{p} \\ \text{s.t.} & A^T\vec{p} \ge \vec{c} \\ & \vec{p} \ge \vec{0} \end{array}\right\}$$

Theorem: Weak duality

Given any solutions $\vec{x}_*$ to primal and $\vec{p}_*$ to dual, it holds that

$$\vec{c}^T\vec{x}_* \le \vec{b}^T\vec{p}_*$$

Proof: $\vec{x}_*$ solution $\Rightarrow$ $\vec{x}_*$ feasible $\Rightarrow$ $A\vec{x}_* \le \vec{b}$.

$\vec{p}_*$ " to dual $\Rightarrow$ $\vec{p}_*$ feasible in dual $\Rightarrow$ $A^T\vec{p}_* \ge \vec{c}$

Also, $\vec{p}_* \ge 0$, so $\vec{p}_*^T A\vec{x}_* \le \vec{p}_*^T\vec{b} \Rightarrow$

$\Rightarrow \vec{x}_*^T A^T\vec{p}_* \le \vec{b}^T\vec{p}_* \Rightarrow \vec{x}_*^T\vec{c} \le \vec{x}_*^T A^T\vec{p}_* \le \vec{b}^T\vec{p}_* \Rightarrow$

$\Rightarrow \vec{c}^T\vec{x}_* \le \vec{b}^T\vec{p}_*$ //

Theorem: Strong duality.

If $\vec{x}_*$ optimum solution to primal, with $\vec{c}^T\vec{x}_*$ finite, then,

$$\vec{c}^T\vec{x}_* = \vec{b}^T\vec{p}_* .$$

• Example: $\max\ 3x+2y$ s.t. $x+y\le 5$, $x,y\ge 0$ $\xrightarrow{\text{dual}}$ $\min\ 5p$ s.t. $p\ge 3$, $p\ge 2$, $p\ge 0$

$$\left.\begin{array}{l}\max\ 3x+2y\\ \text{s.t.}\quad x+y\le 5\\ \qquad x,y\ge 0\end{array}\right\} \xrightarrow{\text{dual}} \left\{\begin{array}{l}\min\ 5p\\ \text{s.t.}\quad p\ge 3\\ \qquad\ p\ge 2\\ \quad p\ge 0\end{array}\right\}$$

Trivially, solution to dual is $p_*=3$, $\vec{b}^T\vec{p}_*=15$, so we have that $\vec{c}^T\vec{x}_*=15$.

That is, the optimal solution to primal satisfies $3x+2y=15$.

Thus, $x = 5-\frac{2}{3}y$

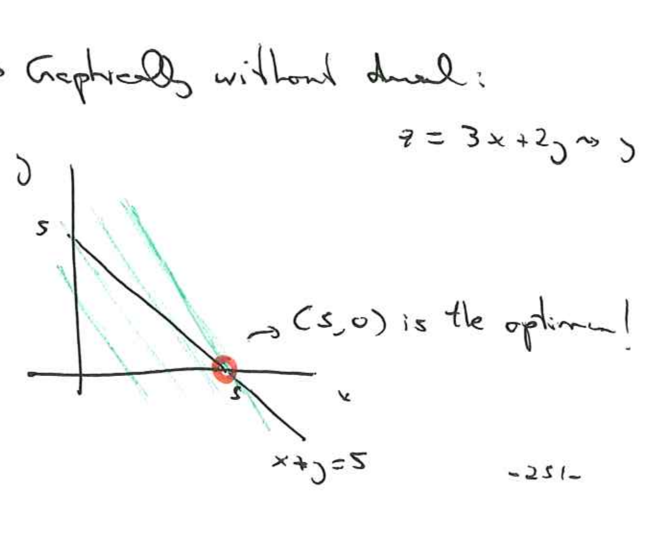
$\hookrightarrow x+y\le 5 \rightarrow 5+\frac{y}{3}\le 5 \rightarrow y\le 0 \Rightarrow y=0 \Rightarrow x=5$.

optimum $\left.\begin{array}{l}x_*=5\\ y_*=0\end{array}\right\|$

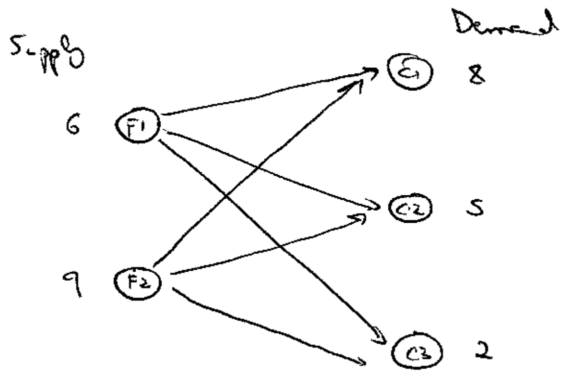
→ Graphically without dual:

$z = 3x+2y \rightsquigarrow y = \frac{z}{2}-\frac{3}{2}x$

slope bigger than -1.



Example: Write the primal standard form and its dual for the following transport problem:



Costs of transport

	C1	C2	C3
F1	5	5	3
F2	6	4	1

$\} \; c_{ij}$

Goal: Minimize cost of satisfying all the demand.

Denote x_{ij} = product transported from factory i to center j $\left(\begin{array}{l} i=1,2 \\ j=1,2,3 \end{array}\right)$.

Then, we want to

$$\min \sum_{i,j} c_{ij} x_{ij} = 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + x_{23}$$

Constraints:

$$\left.\begin{array}{l} x_{11} + x_{21} = 8 \\ x_{12} + x_{22} = 5 \\ x_{13} + x_{23} = 2 \end{array}\right\} \text{ satisfying the demand.}$$

$$\left.\begin{array}{l} x_{11} + x_{12} + x_{13} \leq 6 \\ x_{21} + x_{22} + x_{23} \leq 9 \end{array}\right\} \text{ the production of each factory is limited,}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

This is not in standard form: we have min instead of max, and some = instead of $\leq$.

1) $\min \vec{c}^T \vec{x} = -\max\left(-\vec{c}^T\vec{x}\right)$

2) Ex: $x_{11} + x_{21} = 8 \rightsquigarrow$ equivalent to $\left.\begin{array}{l} x_{11} + x_{21} \leq 8 \\ x_{11} + x_{21} \geq 8 \end{array}\right\}$

and $x_{11} + x_{21} \geq 8$ is same as $-x_{11} - x_{21} \leq -8$.

So, replace $x_{11} + x_{21} = 8$ by $\left.\begin{array}{l} x_{11} + x_{21} \leq 8 \\ -x_{11} - x_{21} \leq -8 \end{array}\right\}$.

By doing so, the problem in standard form is

$$\left.\begin{array}{l} -\max\left(-\vec{c}^T\vec{x}\right) \\ \text{s.t. } A\vec{x} \leq \vec{b} \\ \vec{x} \geq \vec{0} \end{array}\right\} \quad \text{with } \vec{c}^T = \begin{bmatrix} 5 & 5 & 3 & 6 & 4 & 1 \end{bmatrix}$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}, \quad \vec{b} = \begin{bmatrix} 8 \\ -8 \\ 5 \\ -5 \\ 2 \\ -2 \\ 6 \\ 9 \end{bmatrix}$$

Then, its dual is

$$-\min \ \vec{b}^T \vec{p} \quad \text{s.t.} \quad \left. \begin{array}{l} A^T \vec{p} \geq \vec{c} \\ \vec{p} \geq \vec{0} \end{array} \right\}$$

- <u>Introduction to Simplex Method</u>

$$\left. \begin{array}{l} \max \ \vec{c}^T \vec{x} \\ \text{s.t.} \ A\vec{x} \leq \vec{b} \\ \qquad \vec{x} \geq \vec{0} \end{array} \right\}$$

We'll follow a particular example:

$$\left. \begin{array}{l} \max \ 3x - 4y \\ \text{s.t.} \quad x + y \leq 10 \\ \qquad 3x + 2y \leq 6 \\ \qquad x, y \geq 0 \end{array} \right\}$$

1) Step 1): Write the problem in standard form.

2) Step 2): Add slack variable so that inequalities become equalities.

$$\left. \begin{array}{l} \max \ 3x - 4y \\ \text{s.t.} \quad x + y + s_1 = 10 \\ \qquad 3x + 2y + s_2 = 6 \\ \qquad x, y, s_1, s_2 \geq 0 \end{array} \right\}$$

3) Create the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -\vec{c}^T & 0 & 0 \\ 0 & A & I_2 & \vec{b} \end{array} \right] = B \rightsquigarrow \left| \begin{array}{l} z = \vec{c}^T \vec{x} \\ A\vec{x} + \vec{s} I = \vec{b} \end{array} \right.$$

In the ex.,

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & x & y & s_1 & s_2 & \end{array} \\
B = & \left[\begin{array}{ccccc|c} 1 & -3 & 4 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 10 \\ 0 & 3 & 2 & 0 & 1 & 6 \end{array}\right]
\end{array}
$$

What we are doing is: fix x, y to zero, and slack variables to $\vec{b}$.
(they are pivots)

<u>Remark</u>: If we start with the <u>standard form</u>, we will always have this structure!

4) <u>Pivoting</u>: Look at the first row.

If one entry is negative, it means that by increasing the corresponding variable we increase the profit z.

→ So we want x to be a new pivot.

→ Q: Should we choose 2^{nd} or 3^{rd} row?

↳ $\left.\begin{array}{l} x + y + s_1 = 10 \\ 3x + 2y + s_2 = 6 \end{array}\right\}$ → $\left.\begin{array}{l} x + s_1 = 10 \\ 3x + s_2 = 6 \end{array}\right\}$ → If we pivot in the second row, s_1 will stop being a pivot, and thus $s_1 = 0$. This means $x = 10$.

as $y = 0$ (no pivot).

$s_1 = 10$
$s_2 = 6$

If we pivot in the third, $s_2 = 0 \rightsquigarrow x = 2$

Notice that the second equation $x + y + s_1 = 10$ tell us that $x \leq 10$,

while the third $3x + 2y + s_2 = 6 \rightsquigarrow x \leq 2$.

So the third is more restrictive.

→ We always choose the row with minimum quotient b_i / a_{ij}
(when introducing pivot in col j).

So,

$$\begin{bmatrix} 1 & -3 & 4 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \vdots & 10 \\ 0 & \boxed{3} & 2 & 0 & 1 & \vdots & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 6 & 0 & 1 & \vdots & 6 \\ 0 & 0 & 1/3 & 1 & -1/3 & \vdots & 8 \\ 0 & 1 & 2/3 & 0 & 1/3 & \vdots & 2 \end{bmatrix}$$

Now we have, $x = 2,\ s_1 = 8,\ y = 0 = s_2$ ‖ optimum! (first row of B ≥ 0).
$z = 6 - 6y - s_2 = 6$

this tell us any modification will make z smaller.