MATH 312
LECTURE 22 : Graphs ($\sim$ Chap. 10)

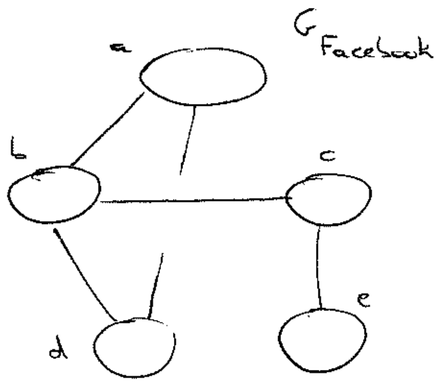
A graph is a collection of nodes (or vertices) and edges between the nodes.

- We will distinguish three types:

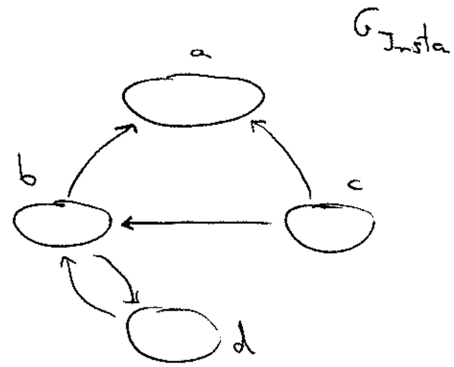
1) ~~Un~~Directed graphs
2) Directed graphs
3) Weighted graphs

<u>Example:</u>

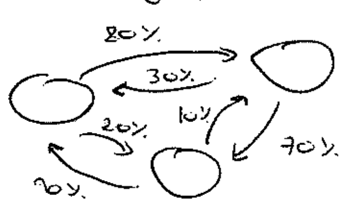
1) Undirected graphs



2) Directed graphs



3) Weighted graphs



• Using matrices to describe graphs.

Adjacency matrix

1) For undirected graphs: $a_{ij} = \begin{cases} 1 & \text{if } \exists \text{ edge between nodes } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$

2) For directed graphs: $a_{ij} = \begin{cases} 1 & \text{if } \exists \text{ arrow from node } i \text{ to node } j \\ 0 & \text{otherwise} \end{cases}$

Example:

For $G_{Face.} \longrightarrow$ $A = \begin{array}{cc} \begin{array}{ccccc} a & b & c & d & e \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array}$

<u>Remark</u>: For undirected graphs, the adjacency matrix is always symmetric.

<u>Remark</u>: The sum of the ith column tells how many edges touch the node i. This is called the <u>degree of node i</u>.

↳ In the example, it represents the n° of friends (in Facebook) of person i.

$\sum_{j=1}^{n} a_{ji}$

<u>Question</u>: What is the meaning of A^2?

Call $B = A^2$, that is, $b_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$.

From the definition of a_{ik}, a_{kj}, we see that $a_{ik}a_{kj}$ is 1 only if both a_{ik} and a_{kj} are 1 (otherwise zero). That is there is a path from node i to node k and from node k to node j.

So, b_{ij} counts the n° of paths of <u>length 2</u> connecting nodes i and j.

In G_{Facebook}, one can check that

$$A^2 = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}$$

(Of course, A^2 has to be symmetric too).

<u>Remark</u>: The sum of column ith, minus the diagonal entry a_{ii}, tells how many people are connected to person i through a friend of i.

Example.

For $G_{Insta} \longrightarrow A = \begin{array}{c c} \begin{array}{cccc} a & b & c & d \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$

Remarks: 1) The matrix is not symmetric (in general).

(Of course: I can follow you even if you don't follow me).

2) Sum of column i = n° of arrows going into node i

↳ Called "In degree"

↳ In the example ≡ n° of followers of person ith.

3) Sum of row i = n° of arrows going out of node i.

↳ Called "Out degree"

↳ In the ex. ≡ n° of accounts person i is following.

Question: Meaning of A^2? (exercise...)

Incidence Matrix

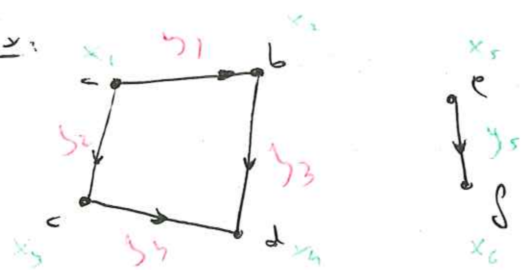
Mostly used for directed graphs.

We assign a column to each node and a row to each edge: so it is generally not symmetric.

It is defined as:

$$a_{ij} = \begin{cases} -1 & \text{if } \cancel{\text{edge}} \text{ arrow } i \text{ starts at node } j \text{ (node called source)} \\ 1 & \text{if arrow } i \text{ ends at node } j \text{ (node called sink)} \\ 0 & \text{otherwise} \end{cases}$$

Ex:



$$\begin{array}{c} \\ ab \\ ac \\ bd \\ cd \\ ef \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{cccccc} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right] \end{array} = I_1$$

→ N° of edges = N° of rows ≡ m

→ N° of nodes = N° of cols ≡ n

Remark: Each row adds up to 0.

• Application of the incidence matrix: Counting connected components and nº of loops.

Let's see how. Let's study the nullspace and left nullspace of the matrix I.

<u>Nullspace, N(I)</u>

$$\begin{array}{c|c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} ab \\ ac \\ bd \\ cd \\ ef \end{array} & \begin{bmatrix} -1 & 1 & & & & \\ -1 & & 1 & & & \\ & -1 & & 1 & & \\ & & -1 & 1 & & \\ & & & & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \\ x_6 - x_5 \end{bmatrix}$$

When is that equal to zero?

You can check $\rightarrow$ $x_2 = x_1 = x_3 = x_4$

$x_5 = x_6$ $\rightarrow$ Basis for $N(I) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

4 independent equations
$n = 6$ $\Big\} \rightarrow \dim N(I) = 2$

So, $\| \text{nº of connected components} = \dim N(I) \|$

Moreover, each vector in the basis of dim N(I) describes each connected component.

Left nullspace, $N(I^T)$

Using equations:

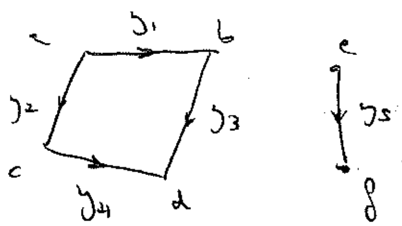
$$\vec{y}^T I = \vec{0} \rightsquigarrow [y_1 \; y_2 \; y_3 \; y_4 \; y_5] \begin{bmatrix} -1 & 1 & & & & \\ -1 & & 1 & & & \\ & -1 & & 1 & & \\ & & -1 & 1 & & \\ & & & & -1 & 1 \end{bmatrix} = [0 \; 0 \; 0 \; 0 \; 0] \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} -y_1 - y_2 = 0 \\ y_1 - y_3 = 0 \\ y_2 - y_4 = 0 \\ y_3 + y_4 = 0 \\ -y_5 = 0 \\ y_5 = 0 \end{array}\right\}$$

If we think of the y_i as currents going along the edges with sign giving by the arrow, these is telling us that:

"current going in = current going out" for each node

↳ Kirchhoff's currents law.



Moreover, let's see vectors in $N(I^T)$.

First, $\dim N(I) = 2 =$ n° of connected components, so

$r = n - 2 = 6 - 2 = 4 \Rightarrow \dim N(I^T) = m - 4 = 5 - 4 = 1$.

So a basis for $N(J^T)$ will only contain 1 vector.

From the equations, we find

$$\text{Basis of } N(J^T) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

‖ It correspond to a current flowing around the loop abcd

This is always the case:

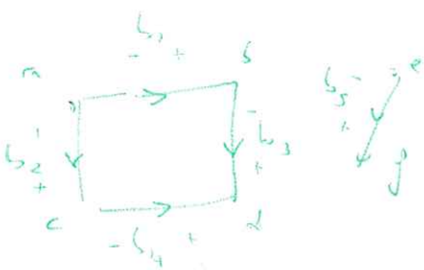
‖ $\dim N(J^T) = n^o$ of (independent) loops

‖ Each vector in the basis of $N(J^T)$ describes a loop.

We can see it from the matrix:

Loop ab, bd, cd, ac → ab + bd − cd − ac

$$\begin{array}{l} ab \\ bd \\ -cd \\ -ac \end{array} \quad \begin{array}{cccccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{array}$$



→ Column space $C(J)$: Kirchhoff's voltage law.

↳ $Jx = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \\ x_6 - x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$ Conditions on the b's to be in $C(J)$?

↳ Notice that if you add up around a circle you obtain zeros. ‖

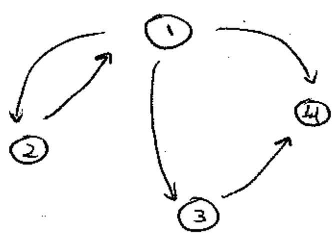
- Measuring the importance of nodes: <u>Centrality</u>

Many options: ex → use degree of each node.

→ use difference between "in degree" and "out degree".

→ give more importance to arrows coming from important nodes... (marketing, networks, ...)

<u>PageRank</u>

G 

Iterative algorithm: Importance of each node is determined by the number of edges going in weighted by the importance of those nodes.

In particular,

the weights are: $\frac{1}{\text{n}^{\text{o}} \text{ of links going out}}$

$$
\left.\begin{aligned}
p_1(k+1) &= p_2(k)\frac{1}{2} \\
p_2(k+1) &= p_1(k)\frac{1}{3} + p_2(k)\frac{1}{2} \\
p_3(k+1) &= p_1(k)\frac{1}{3} \\
p_4(k+1) &= p_1(k)\frac{1}{3} + p_3(k)
\end{aligned}\right\} \rightarrow
$$

In matrix form:

$$\vec{p}(k+1) = \underbrace{\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \end{bmatrix}}\vec{p}(k)$$

a_{ij} = "probability" of going from node j to i

→ Transition matrix (A)
(weighted adjacency matrix)

In general,

$$A \rightarrow \quad a_{ij} = \begin{cases} \frac{1}{n_j} & \text{if there is an arrow from } j \text{ to } i, \\ & \text{with } n_j = \text{nº of arrows going out of node } j. \\ 0 & \text{otherwise} \end{cases}$$

Problems: We cannot ensure that there is a solution or that it is unique.

↳ We would like to use the consequence of Perron-Frobenius for Markov positive matrix:

Th: A positive Markov matrix always has $\lambda = 1$ non-repeated, $|\lambda| < 1$ for all others, and therefore it has a unique steady state $u_\infty = A u_\infty$.

→ We need to fix two things: 1) "Dangling" nodes: those without outgoing arrows.

2) Zero entries.

• Solution to 1): Change the ~~ze~~ column of zeros with the column $\begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix}$.

↳ You can think of this as follows: a random "surfer" will go to any website with same probability if there are no links in the current website.

In our example,

$$A_2 = \begin{bmatrix} 0 & 1/2 & 0 & 1/4 \\ 1/3 & 1/2 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 1 & 1/4 \end{bmatrix}$$

<u>Remark</u>: By construction, after this step we always will have a Markov matrix → cols. add up to 1.

• Solution to 2): "Damping" the matrix A:

$$A_\alpha = (1-\alpha)A_2 + \frac{\alpha}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}, \quad \alpha \in [0,1]$$

Meaning: From any website you can go to any other with some small probability.

↳ If $\alpha \approx 1$, we lose the structure of A.