MATH 312
LECTURE 21

: Perron-Frobenius Theorem.

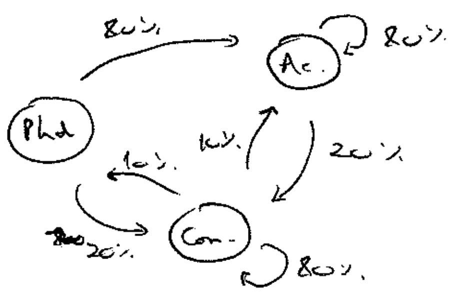
We saw the other day that (discrete) Markov chains can be described through their transition matrix:

$\vec{x}(k) = A\,\vec{x}(k-1)$,

where A is a Markov matrix, i.e. nonnegative entries; columns add up to 1.

<u>Example</u>: Each year, 80% of people doing a phd go to academia, and 20% to a company. 80% of people in academia remain there, while 20% go to a company. Finally, 10% of people in a company leave to academia, 10% move to study a phd, and the rest stay.

1) Write down the state diagram and the transition matrix.



$$\vec{x}(k) = \begin{array}{cl} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} & \begin{array}{l} -\text{phd} \\ -\text{acd.} \\ -\text{comp} \end{array} \end{array}$$

$$P = \begin{bmatrix} 0 & 0 & 0.1 \\ 0.8 & 0.8 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix}$$

• $\vec{x}(k) = A\vec{x}(k-1)$ with initial condition $\vec{x}(0)$.

If A is diagonalizable, the solution is given by

$$\vec{x}(k) = c_1 \lambda_1^k \vec{u}_1 + \dots + c_n \lambda_n^k \vec{u}_n \text{, where}$$

$$\begin{cases} \vec{u}_1, \dots, \vec{u}_n \text{ are the eigenvectors of } A\text{, with eigenvalues } \lambda_1, \dots, \lambda_n. \\ c_1, \dots, c_n \text{ are determined by } \vec{x}(0). \end{cases}$$

(indeed, $\vec{x}(0)\Big|_{\mathcal{U}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, with $\mathcal{U}$ the basis of eigenvectors).

• <u>Important result</u>: If A is Markov with $\lambda = 1$ non repeated eigenvalue and all other satisfy $|\lambda| < 1$, then

$\vec{x}(k) = \vec{u}_1 + c_2 \lambda_2^k \vec{u}_2 + \dots + c_n \lambda_n^k \vec{u}_n$, and therefore (Why?)

$\downarrow$ (<u>theorem</u>)

$\boxed{\vec{x}_\infty = \vec{u}_1}$ does not depend on the initial state.

• <u>Remark</u>: In general, anything can happen:

1) No steady state: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2) Steady state non unique: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ($\vec{x}_\infty$ depends on $\vec{x}(0)$).

- Goal: To find sufficient conditions that guarantee that
  1) $\lambda = 1$ is eigenvalue of A.
  2) $|\lambda| < 1$ for all the rest.

In that case, we can ensure that there exist a unique steady, independent of $\vec{x}(0)$.

- Theorem: If A is Markov with positive entries, then
  1) $\lambda = 1$ eig. of A.
  2) $|\lambda| < 1$ for all other.

We will need to prove first Perron-Frobenius theorem.

From now on, we will say

→ A is positive if all the entries are positive, i.e., $a_{ij} > 0$.

→ $\vec{u}$ " " " " the components " ", i.e., $u_i > 0$

→ $\vec{x} > \vec{y}$ if $x_i > y_i$ for all i

<u>Perron-Frobenius Theorem</u>: Let A $n\times n$ positive matrix (i.e., $a_{ij}>0$).

Then,

1) $\rho(A)$ is a non-repeated eigenvalue of A.

2) All other eigenvalues satisfy $|\lambda| < \rho(A)$.

3) $\rho(A)$ has a positive eigenvector.

4) No other eigenvalue has a positive eigenvector.

We will split the proof in several parts.

<u>Lemma 1</u>: Let $\vec{x}, \vec{y}$ vectors such that $\vec{x} \geq \vec{y}$ but $\vec{x} \neq \vec{y}$, and A positive. Then

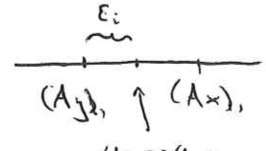
$A\vec{x} > A\vec{y}$. In particular, $\exists\, \varepsilon > 0 \,/\, A\vec{x} > (1+\varepsilon)A\vec{y}$.

<u>Proof</u>: ① $\vec{x} \geq \vec{y} \Rightarrow x_i - y_i \geq 0 \Rightarrow \cancel{A(\vec{x}-\vec{y})}$

$\hookrightarrow$ $\cancel{A(\vec{x}-\vec{y})}$ $(A(\vec{x}-\vec{y}))_i = \sum_{j=1}^{n} a_{ij}(x_j - y_j) \geq$

$\geq (\min_{j=1,\dots,n} a_{ij}) \sum_{j=1}^{n} (x_j - y_j) > 0$ since $\vec{x} \neq \vec{y}$, and $a_{ij} > 0$.

② $A\vec{x} > A\vec{y} \Rightarrow A\vec{x} > (1+\varepsilon)A\vec{y}$:



Take $\varepsilon_i = \frac{1}{2}(A\vec{x} - A\vec{y})_i$ and $\varepsilon = \min_{i=1,\dots,n} \varepsilon_i$ //

• <u>Prop 1</u>: $\rho(A)$ is eigenvalue with a positive eigenvector.

<u>Proofs</u>

Consider the numbers $t \geq 0$ / $A\vec{x} \geq t\vec{x}$ for some nonnegative $\vec{x} \neq \vec{0}$.

Think of the biggest of those numbers, $t_{máx}$: $A\vec{x} \geq t_{máx}\vec{x}$.

That is, if we increase $t_{máx}$, there is no $\vec{x}$ verifying the inequality.

Let's prove that, indeed, for this $t_{máx}$, the equality holds:

Assume the contrary: $\cancel{A\vec{x} > t_{máx}\vec{x}}$ for some $\vec{x} \neq 0$. $\exists i$ / $(A\vec{x})_i > t_{máx} x_i$

Then,

$$A^2\vec{x} > t_{máx} A\vec{x} \overset{\text{Lemma 1}}{\Rightarrow} A(A\vec{x}) > (1+\varepsilon)t_{máx}(A\vec{x}) \rightarrow\!\!\leftarrow .$$

So $A\vec{x} = t_{máx}\vec{x}$, and $t_{máx}$ is eigenvalue.

→ Is $t_{máx}$ the larger eigenvalue? I.e., $\rho(A) = t_{máx}$?

→ Is $\vec{x}$ corresponding to $t_{máx} = \rho(A)$ positive?

↳ Easy: $\left.\begin{matrix}\vec{x} \geq \vec{0} \\ \vec{x} \neq \vec{0}\end{matrix}\right| \underset{\text{Lemma 1}}{\Rightarrow} A\vec{x} > \vec{0}$. Since $t_{máx}\vec{x} = A\vec{x} > \vec{0} \Rightarrow \vec{x} > 0$!

↳ Suppose λ is another eigenvalue with $\vec{u}$ eigenvector.

$A\vec{u} = \lambda\vec{u}$. Then $|\lambda\vec{u}| = |\lambda||\vec{u}| = |A\vec{u}| = |\sum a_{ij}u_j| \leq \sum |a_{ij}u_j| \overset{A \text{ positive}}{=} A|\vec{u}| \Rightarrow$

$\Rightarrow A|\vec{u}| \geq \underbrace{|\lambda|}_{\text{nonnegative.}}|\vec{u}| \Rightarrow |\lambda|$ cannot be bigger than $t_{máx}$.

· **Prop. 2**: $\rho(A)$ is non repeated eigenvalue of A.

Proof: We won't really prove this. We will only prove that its eigenspace has dimension 1, i.e. that $N(A - \rho(A)I) = 1$.

Suppose $\vec{w} > \vec{0}$ is eigvector of $\rho(A)$, and $\vec{u}$ another l.i. eigvector of $\rho(A)$.

Then we can find α such that $\vec{w} - \alpha\vec{u} \geq 0$ with one entry $= 0$.

[Let $\alpha_i = \frac{w_i}{u_i}$ for those $\vec{u}_i \neq 0$. Let $\alpha = \min \alpha_i$.

Then $\vec{w} - \alpha\vec{u}$ has at least one entry zero. Not all zero since $\vec{w}$ and $\vec{u}$ are l.i.]

Using Lemma 1, $A(\vec{w} - \alpha\vec{u}) > 0$, and so

$$A\vec{w} - \alpha A\vec{u} = \rho(A)\vec{w} - \alpha\rho(A)A\vec{u} = \rho(A)(\vec{w} - \alpha\vec{u}) > 0 \rightsquigarrow \vec{w} - \alpha\vec{u} > 0 \quad \cancel{}$$

(at least one entry $= 0$)

Remark: $\rho(A) \neq 0$ due to Prop 1.

Prop 3: All other eig. $|\lambda| < \rho(A)$.

Proof.

First, $|\lambda| \leq \rho(A)$ by def. of $\rho(A)$. So just need to prove that

If $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$ (not negative, not complex).

· Let $\lambda \neq \rho(A)$, $|\lambda| = \rho(A)$, with eigvector $\vec{u}$: $A\vec{u} = \lambda\vec{u}$.

$$\left.\begin{array}{l}\text{Then, } |A\vec{u}| = |\lambda\vec{u}| = \overbrace{|\lambda|}^{=\rho(A)}|\vec{u}| \\ \qquad |A\vec{u}| = |\sum a_{ij}u_j| \leq \sum |a_{ij}u_j| = A|\vec{u}|\end{array}\right\} \to A|\vec{u}| \geq \rho(A)|\vec{u}| \underset{\text{prop 1)}}{\Rightarrow}$$

$$\to A|\vec{u}| = \rho(A)|\vec{u}| > 0$$

Thus,

$$(A|\vec{u}|)_1 = \sum a_{1j}|u_j| = \rho(A)|u_1| > 0 \quad (1)$$

and

$$(A\vec{u})_1 = (\lambda\vec{u})_1 = \sum a_{1j}u_j \Rightarrow |(\lambda\vec{u})_1| = \rho(A)|u_1| = \left|\sum a_{1j}u_j\right| \quad (2)$$

That is, $$\left\| \sum a_{1j}|u_j| = \left|\sum a_{1j}u_j\right| \right\|$$

This implies that

$$a_{11}|u_1| + a_{12}|u_2| + \dots + a_{1n}|u_n| = |a_{11}u_1 + \dots + a_{1n}u_n| \to$$

$$\Rightarrow a_{1j}u_j = \alpha_j a_{11}u_1$$

(*) We are using that for $z_1, \dots, z_n \neq 0$ complex numbers with

$|z_1 + \dots + z_n| = |z_1| + \dots + |z_n|$, then each z_j is a positive multiple of z_1.

Therefore,

$$u_j = \left(\alpha_j \frac{a_{11}}{a_{1j}}\right) u_1 \Rightarrow \vec{u} = \underbrace{\begin{bmatrix} 1 \\ \alpha_2 a_{11}/a_{12} \\ \vdots \\ \alpha_n a_{11}/a_{1n} \end{bmatrix}}_{\vec{w} > 0 \ (\text{notice } \vec{u} = u_1 \vec{w})} u_1$$

• Going back, $\lambda \vec{u} = A\vec{u} \Rightarrow \lambda \vec{w} = A\vec{w} = |A\vec{w}| = |\lambda \vec{w}| = \rho(A)\vec{w} \Rightarrow$

$$\Rightarrow \lambda \vec{w} = \rho(A)\vec{w} \underset{\uparrow}{\Rightarrow} \lambda = \rho(A) /\!\!/ .$$

(first component of $\vec{w}$ is 1)

[Hasta aquí ← in class]

• <u>Prop 4</u>: No other eigenvalue has a positive eigenvector.

<u>Proof</u>: Assume $\lambda \neq \rho(A)$ with $\vec{u} > 0$. Then $A\vec{u} = \lambda \vec{u}$ (1).

We know from Prop 1 that $\rho(A)$ is eigenvalue of A. Let's see it is also of A^T:

No hecho en clase

(Chapt. 5)

<u>Lemma</u>: $\rho(A) = \rho(A^T)$

↳ Indeed, A and A^T have the same eigenvalues! (but different eigenvectors).

$\det(A - \lambda I) = \det((A-\lambda I)^T) = \det(A^T - \lambda I) /\!\!/$

Let $\vec{\omega} > 0$ eigvector corresponding to $\rho(A^T)$ of A^T:

$$A^T \vec{\omega} = \rho(A)\vec{\omega} \quad (2)$$

- Multiply (1) by $\vec{\omega}^T$: $\vec{\omega}^T A \vec{u} = \lambda \vec{\omega}^T \vec{u}$

and transpose: $\vec{u}^T \underbrace{A^T \vec{\omega}}_{\overset{(2)}{=} \rho(A)\vec{\omega}} = \lambda \vec{\omega}^T \vec{u} \Rightarrow \rho(A)\vec{\omega}^T \vec{u} = \lambda \vec{\omega}^T \vec{u} \Rightarrow$

$$\Rightarrow (\rho(A) - \lambda)\, \underbrace{\vec{\omega}^T \vec{u}}_{\substack{(\vec{\omega}^T\vec{u} > 0 \text{ since both} \\ \text{are } > 0)}} = 0 \Rightarrow \rho(A) = \lambda. \;\; \not\mathrel{\rightarrow}\!\!\leftarrow.$$

We did this in class

- We can now prove that for A Markov with positive entries, then

$$\begin{cases} \lambda = 1 \text{ is eig. of } A \text{ (with positive eigenvector} \rightsquigarrow \text{steady state)}. \\ |\lambda| < 1 \text{ for all other} \end{cases}$$

<u>Proof:</u> We just need to prove that $\rho(A) = 1$.

$\rightarrow$ <u>$\rho(A) \leq 1$</u>: ~~Let λ eig.: $A\vec{v} = \lambda\vec{v}$ ($\vec{v} \neq \vec{0}$)~~ Suppose $\rho(A) > 1$.

Then, pick the corresponding positive eigvector $\vec{u} > 0$: $A\vec{u} = \rho(A)\vec{u}$

We can scale $\vec{u}$ so that their components add up to 1, but then $A\vec{u}$ also has components that add up to 1, while $\rho(A)\vec{u}$ would add up to $\rho(A) > 1$ $\not\mathrel{\rightarrow}\!\!\leftarrow$.

Now, we can check that $\rho(A)=1$ is always an eigenvalue for A Markov.

Recall $\rho(A)=\rho(A^T)$ and notice that

$$A^T\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}=\begin{bmatrix}\text{sum of row 1 of } A^T\\ \vdots\\ \text{sum of row } n \text{ of } A^T\end{bmatrix}=\begin{bmatrix}\text{sum of col 1 of } A\\ \vdots\\ \text{sum of col } n \text{ of } A\end{bmatrix}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\Rightarrow$$

$\Rightarrow \lambda=1$ is eigenvalue of A^T, and thus of A //.

No work in Class.

Questions:

~~1) If A has cols. that add up to 1, then $\lambda=1$ is always an eigenvalue.~~
~~↳ True~~

1) A, A^T have same eigvalue and eigvectors.

~~2) A Markov matrix with some entries equal to zero cannot have a unique steady~~

2) If a Markov chain is given by A matrix with some zero entries, then the steady state depends on $\vec{x}(0)$.

3) For $\vec{x}(k)=A\vec{x}(k-1)$, with $\vec{x}(0)$, and A diagonalizable the solution is

$u=\{\vec{u}_1,\dots,\vec{u}_n\}$ eigvectors of A.

$\vec{x}(k)=c_1\lambda_1^k\vec{u}_1+\dots+c_n\lambda_n^k\vec{u}_n$, where $\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}=\vec{x}(0)/_u$

For A positive Markov and $\vec{u}_1$ eigvector with $\lambda=1$, can we pick $\vec{x}(0)$ such that $c_1=0$?

4) A matrix whose cols. add up to 1 always has $\lambda=1$ as eig. (even if some entries are zero).