Homework 11

$\boxed{1}$

To be a basis, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ need to be linearly independent and need to span the vector space of all parabolas $a+bx+cx^2$.

- Lin. indep: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \iff c_1 = c_2 = c_3 = 0$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1 m_1 + c_2 m_4 + c_3 m_7 + (c_1 m_2 + c_2 m_5 + c_3 m_8)x +$$
$$+ (c_1 m_3 + c_2 m_6 + c_3 m_9)x^2 = 0 \iff$$

$$\iff \begin{array}{l} c_1 m_1 + c_2 m_4 + c_3 m_7 = 0 \\ c_1 m_2 + c_2 m_5 + c_3 m_8 = 0 \\ c_1 m_3 + c_2 m_6 + c_3 m_9 = 0 \end{array} \iff \underbrace{\begin{bmatrix} m_1 & m_4 & m_7 \\ m_2 & m_5 & m_8 \\ m_3 & m_6 & m_9 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To ensure that $c_1 = c_2 = c_3$, the above system has to have a unique solution. That is, the nullspace of A has to be the trivial one. So A has to non singular $\Rightarrow \det(A) \neq 0$.

- Span: For every $\vec{b} = a + bx + cx^2$, there exist c_1, c_2, c_3 / $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$. I.e., the system $A\vec{c} = \vec{b}$ needs to have a solution for every $\vec{b}$. We need thus 3 pivots. Therefore, A is not singular $\rightarrow \det(A) \neq 0$.

• In conclusion, if $\det\begin{pmatrix} m_1 & m_4 & m_7 \\ m_2 & m_5 & m_8 \\ m_3 & m_6 & m_9 \end{pmatrix} \neq 0$, then

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for the vector space of parabolas $a + bx + cx^2$.

• <u>Remark</u>: We could have started choosing a standard basis $\vec{e}_1 = 1$, $\vec{e}_2 = x$, $\vec{e}_3 = x^2$, so that we can write in this basis

$$\vec{v}_1 = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} m_4 \\ m_5 \\ m_6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} m_7 \\ m_8 \\ m_9 \end{bmatrix}$$

(more exactly, if $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, then $\vec{v}_1|_{\mathcal{E}} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}, \ldots$).

And now we can work as usual with column vectors.

$\boxed{2}$ $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, $A^{\infty} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$

$$\det(A - \lambda I) = \frac{1}{100}(8-10\lambda)(7-10\lambda) - \frac{2\cdot 3}{100} = 0 \Leftrightarrow 56 - 150\lambda + 100\lambda^2 - 6 = 0 \Leftrightarrow$$

$$\Leftrightarrow 2\lambda^2 - 3\lambda + 1 = 0 \Leftrightarrow \lambda = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} \Rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1/2. \end{matrix}$$

Eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$A - \tfrac{1}{2} I = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix} \to \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since A is diagonalizable, we can write $A = SDS^{-1}$, where

$S = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$. Thus,

$$A^k = SD^kS^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2^k \end{bmatrix}\frac{-1}{5}\begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} \Rightarrow$$

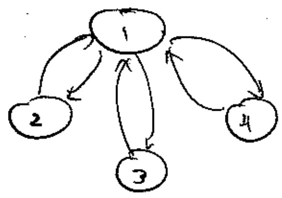
$$\Rightarrow \lim_{k\to\infty} A^k = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\frac{1}{5} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}\frac{1}{5} =$$

$$= \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix} //$$

direction of $\vec{v}_1$

[Since $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, only the part of A corresponding to $\lambda_1 = 1$ "survives" after many iterations].

$\boxed{3}$



a) $$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) $$A \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} = \frac{1}{4} A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 9 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^2 \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} = \frac{1}{12} A \begin{bmatrix} 9 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}$$

It's clear that we are oscillating between $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} / 4$ and $\begin{bmatrix} 9 \\ 1 \\ 1 \\ 1 \end{bmatrix} / 12$.

c) We can compute the eigenvalues of A:

$\text{rank}(A) = 2 \Rightarrow \dim N(A) = 2 \Rightarrow \lambda_1 = 0 = \lambda_2$.

Since A has columns adding up to 1 $\Rightarrow \lambda_3 = 1$.

Finally, $\text{trace}(A) = 0 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \Rightarrow \lambda_4 = -1$.

Having $\lambda_4 = -1$ produces the oscillations.

[Think of a diagonalizable matrix A. Then the solution

to $\left.\begin{array}{l} \vec{x}(k+1) = A\vec{x}(k) \\ \vec{x}(0) = \vec{x}_0 \end{array}\right\}$ is $\vec{x}(k) = A^k \vec{x}_0 = S D^k S^{-1} \vec{x}_0 =$

$$= \underbrace{\begin{bmatrix} \vec{u}_1 | \cdots | \vec{u}_n \end{bmatrix}}_{\text{eigenvectors}} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \vec{c} = c_1 \lambda_1^k \vec{u}_1 + \dots + c_n \lambda_n^k \vec{u}_n$$

$\vec{c} = \vec{x}_0 |_S$ (coordinates of $\vec{x}_0$ in the basis of eigenvectors S)

If $|\lambda| < 1 \Rightarrow \lambda^k \to 0$

$\lambda = -1 \Rightarrow \lambda^k = (-1)^k \to$ oscillating part.

$\lambda = 1 \to \lambda^k = 1 \to$ steady part $\quad \|$

d) $B = 0.85\,A + \frac{0.15}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$B = \frac{17}{60}\begin{bmatrix} 0 & 3 & 3 & 3 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \frac{3}{80}\begin{bmatrix} 1 & \cdots & & 1 \\ \vdots & \ddots & & \vdots \\ & & \ddots & \\ 1 & \cdots & & 1 \end{bmatrix} = \begin{bmatrix} 3/80 & 71/80 & 71/80 & 71/80 \\ 77/240 & 3/80 & 3/80 & 3/80 \\ 77/240 & 3/80 & 3/80 & 3/80 \\ 77/240 & 3/80 & 3/80 & 3/80 \end{bmatrix}$$

e) $\vec{p} = B\vec{p} \rightsquigarrow$ eigvector of B corresponding to $\lambda = 1$.

$$B - I = \begin{bmatrix} -77/80 & 71/80 & 71/80 & 71/80 \\ 77/240 & -77/80 & 3/80 & 3/80 \\ 77/240 & 3/80 & -77/80 & 3/80 \\ 77/240 & 3/80 & 3/80 & -77/80 \end{bmatrix} \to \vec{v} = \begin{bmatrix} 213 \\ 77 \\ 77 \\ 77 \end{bmatrix}$$

So
$$p_1 = 213/444 \approx 0.48$$
$$p_2 = 77/444 = 0.17\bar{3}$$
$$p_3 = 77/444 = 0.173$$
$$p_4 = 77/444 = 0.173$$

$\boxed{P3}$

a) $$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

b) $$A - I = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \dim N(A-I) = 2.$$

So the solution to $\vec{p} = A\vec{p}$ is not unique.

c) $$B = 0.85\,A + \frac{0.15}{5}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 3 & 88 & 3 & 3 & 3 \\ 88 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 88 \\ 3 & 3 & 88 & 3 & 3 \\ 3 & 3 & 3 & 88 & 3 \end{bmatrix} / 100$$

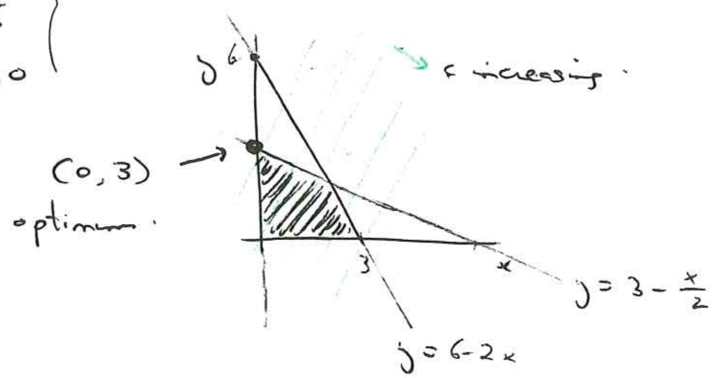
$$\left(\begin{array}{l} 0.85 = 1 - \frac{3}{20} = \frac{17}{20} = 85/100 \\ 0.15/5 = \frac{3}{100} \end{array} \right)$$

$$B - I = \frac{1}{100}(B - 100I) = \frac{1}{100}\begin{bmatrix} -97 & 88 & 3 & 3 & 3 \\ 88 & -97 & 3 & 3 & 3 \\ 3 & 3 & -97 & 3 & 88 \\ & & \cdots & & \end{bmatrix} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So $$\vec{p} = \vec{v}_1 / 5 = \begin{bmatrix} 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \\ 0.20 \end{bmatrix}$$

P4

$$\left.\begin{array}{l} x + 2y \leq 6 \\ 2x + y \leq 6 \\ x \geq 0,\ y \geq 0 \end{array}\right\} \quad \min 2x - y? \rightarrow c = 2x - y \rightarrow y = 2x - c$$



We can checked that (0,3) is the point with minimum cost:

$(0,0) \rightarrow c = 0$

$(0,3) \rightarrow \boxed{c = -3}$

$(3,0) \rightarrow c = 6$

$(2,2) \rightarrow c = 2$

P5

Protein: $12x_1 + 24.9x_2 + 0.1x_3 + 6x_4 + 3x_5 \geq 70$

Calories $246x_1 + 423x_2 + 793x_3 + 93x_4 + 26x_5 \geq 3000$

Calc. $0.1x_1 + 0.2x_2 + 0.03x_3 + 0.05x_4 + 0.1x_5 \geq 1$

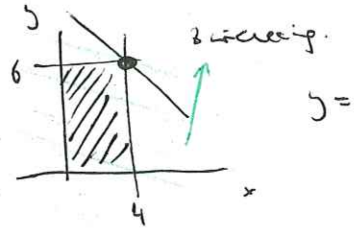
Iron: $3.2x_1 + 0.3x_2 + 2.3x_4 + 2.x_5 \geq 12$

Cost: $0.5x_1 + 2x_2 + x_3 + 0.25x_4 + 0.25x_5$ (minimize it).

P6

$$\left.\begin{array}{ll} \max & z = 2x+3y \\ \text{s.t} & x \le 4 \\ & x+y \le 10 \\ & y \le 6 \\ & x \ge 0,\ y \ge 0 \end{array}\right\}$$

1)



$y = \frac{2}{3} - \frac{2}{3}x$

2) $(0,0),\ (0,6),\ (4,0),\ (4,6)$

3) $f(0,0)=0,\ f(0,6)=18,\ f(4,0)=8,\ \underbrace{f(4,6)=26}$

maximum (at $x=4,\ y=6$).

P7

1)

$$\left.\begin{array}{ll} \max & +2x+3y \\ \text{s.t.} & A\vec{x} \le \vec{b} \\ & \vec{x} \ge \vec{0} \end{array}\right\} \quad \vec{c} = \begin{bmatrix} +2 \\ +3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix}$$

2)

$$B = \begin{array}{c} \begin{array}{cccccccc} & x & y & s_1 & s_2 & s_3 & b \end{array} \\ \left[\begin{array}{c|cc|ccc|c} 1 & -2 & -3 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 1 & 6 \end{array}\right] \end{array}$$

$\sim z = 2x+3y$

$x + s_1 = 4$

$\sim x+y+s_2 = 10$

$y + s_3 = 6$

3) On part 2), we have the matrix B corresponding to the initial point $x = y = 0$.

The variables s_1, s_2, s_3 are pivots with values $\begin{array}{l} s_1 = 4 \\ s_2 = 10 \\ s_3 = 6 \end{array}$.

Looking at the first row, there are negative values so we have to continue.

<u>Iteration 1:</u>

1.1) Choose pivot column: The one with more negative coefficient in row one.

↳ In this case, column 3 (corresponding to y).

1.2) Choose pivot row: The one which minimizes b_i / a_{ij} - with j being the pivot column.

↳ Here we have: for row 3 → $10/1$, row 4 → $6/1$ $\Big\}$ → pick row 4.

1.3) Row reduce: $R1' = R1 + 3R4$, $R3' = R3 - R4$

$$\begin{array}{c} \left[\begin{array}{c|cc|ccc|c} 1 & -2 & 0 & 0 & 0 & 3 & 18 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 & 6 \end{array}\right] \\ \begin{array}{ccccccc} & \uparrow \text{ "free"} & & & & \uparrow \text{ "free"} & \end{array} \end{array}$$

Now y is a pivot variable:

$x = 0$, $y = 6$

$s_1 = 4$, $s_2 = 4$, $s_3 = 0$

- Since in row 1 there is still a negative entry, we continue.

<u>Iteration 2:</u>

1.1) Pivot column: column 2.

1.2) Pivot row: row 2.

1.3) $R1'' = R1' + 2R2$

$$\left[\begin{array}{c|cc|ccc|c} 1 & 0 & 0 & 2 & 0 & 3 & 26 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 & 6 \end{array}\right] \rightarrow \boxed{\begin{array}{l} z = 26 \\ x = 4,\ y = 6 \end{array}}$$

(We stop since first row has no negative values).