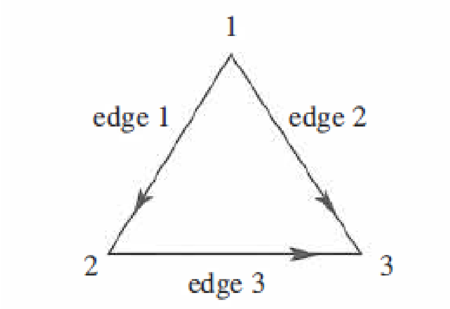
HOMEWORK ASSIGNMENT 10 SOLUTIONS

Name: **Due:** Tuesday December 4, 4pm



Problem 1: Strang 10.1 #1, page 459

Solution: The matrix is

$$Inc = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

The vectors in the null space of Inc are the span of vectors coming from the connected component (each such vector has a 1 in the entry corresponding to each node that is in the component and a 0 for each node that is not in the connected component). In our case, there is one connected component, and the nodes in it are nodes 1,2, and 3. So

$$N(Inc) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

There are a couple ways to see that (1,0,0) is not in the row space. Firstly, the row space $C(Inc^T)$ is orthogonal to $N(Inc)$, but (1,0,0) is not orthogonal to (1,1,1), so it's not in the row space. Secondly, every row of any incidence matrix has entries summing to 0. But the entries of (1,0,0) do not sum to 0, so it is not in the row space.

Problem 2: Strang 10.1 #2, page 459

The transpose matrix is

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

We know each (independent) loop in the graph gives a basis vector of the left null space. We find there is one independent loop: forward along edge 1, forward along edge 3, and then backwards along edge 2. This corresponds to

$$\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

If we view the entries of $\vec{y}$ as currents on edge 1,2,3 respectively, then these voltages satisfy Kirchoff's Current Law.

Problem 3: Strang 10.1 #5, page 460

Such vectors $\vec{f}$ are exactly the vectors in the row space of A. One example of this is $\vec{f} = (0,0,0)$. A non example is $\vec{f} = (1,0,0)$, since the entries do not sum to zero. Since the row space $C(A^T)$ is the orthogonal complement of $N(A)$, which as we calculated in problem 1 is the span of $(1,1,1)$, vectors in $C(A^T)$ are exactly the vectors whose dot product with $(1,1,1)$ is zero.

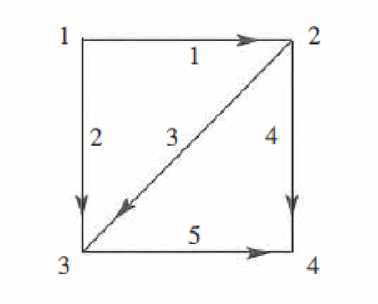
Problem 4: Strang 10.1 #9, page 460

The condition for b_1 to equal the difference $x_2 - x_1$ et cetera is equivalent to the vector $\vec{b} = (b_1, ..., b_5)$ being in the column space of the incidence matrix for this graph.

We also know that a vector $\vec{b}$ is in the column space if and only if it's orthogonal two every vector in $N(Inc^T)$, but $N(Inc^T)$ is generated by two vectors coming from the loops of the graph.

The loop [edge 1 - edge 3 - edge 2] (top left loop) corresponds to the vector $(1,-1,-1,0,0)$ and the loop [edge 3 + edge 5 - edge 4] (bottom right loop) corresponds to the vector $(0,0,1,-1,1)$.

Orthogonality to the first vector gives the requirement that $b_1 - b_2 - b_3 = 0$. Orthogonality to the second vector gives the requirement that $b_3 - b_4 - b_5 = 0$. Since the entries of $\vec{b}$ are voltage differences, this gives Kirchoff's Voltage Law around the two loops in the graph.



Problem 5

a) For an incidence matrix, the number of nodes equals the number of columns. In our case, there are 6 columns, so there are 6 nodes.

The number of edges equals the number of rows. In our case, there are 5 rows, so there are 5 edges.

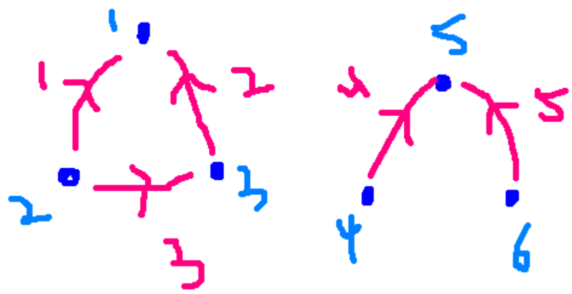
b) We know the number of connected components equals the dimension of $N(A)$. We calculate

$$RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has 4 pivot variables and 2 free variables, so $N(A)$ is dimension 2 and there are two connected components. We even see that the special solution basis for the null space is $(1,1,1,0,0,0), (0,0,0,1,1,1)$, which tells us that nodes 1 2 and 3 are in the first component, and 4,5,6 are in the second component.

Using the fundamental theorem of linear algebra, we can see that dim $N(A^T) = 1$, so there is 1 loop (by calculating a basis for $N(A^T)$, you can see which edges are in the loop and in which direction).

c) You start by drawing 6 nodes and labeling them. Then, you go down the rows of A and for each row of A, you draw an edge that starts at the node where there's a -1, and ends at the node where there's a 1.



Problem 6

The following matrix is the adjacency matrix of some graph:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a) For an adjacency matrix, the number of nodes equals the number of columns (which is equal to the number of rows; an adjacency matrix is square). This matrix has 5 columns so there are 5 nodes.

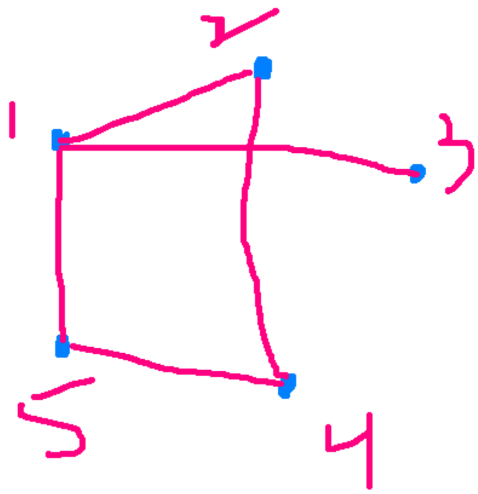
Next, the degree of each node is the sum of the entries of the corresponding row. So the degree of node 1 is 0+1+1+0+1=3, and similarly the degree of node 2 is 2, deg(node 3)=1, deg(node 4)=2, and deg(node 5)=2.

b) We calculate that

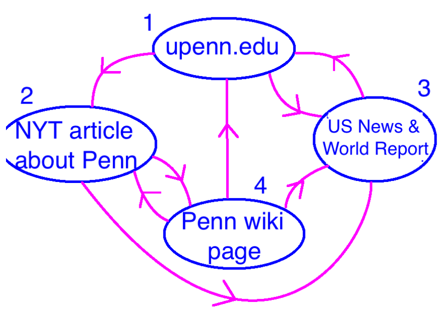
$$A^2 = \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}$$

And note that the number of paths of length 1 from node 1 to node 4 is entry (1,4) of A; this is zero. The number of paths of length 2 from node 1 to node 4 is the (1,4) entry of A^2; this is 2.

c) Again, we start by drawing 5 nodes and connect nodes i and j if and only if there is a 1 in the i, j entry.



Problem 7



a)

$$A = \begin{bmatrix} 0 & 0 & 1 & 1/3 \\ 1/2 & 0 & 0 & 1/3 \\ 1/2 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 0 & 0. \end{bmatrix}$$

b) The PageRank to this graph correspond to the steady state of the Markov chain $\mathbf{p}(k+1) = A\mathbf{p}(k)$, where $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ are the PageRanks of each website. Since the entries of A are not strictly positive, we cannot apply the Perron-Frobenius theorem to ensure the existence of a unique steady state. However, the *nave* PageRank assumes this steady state exists, in which case it is given by $\mathbf{p} = A\mathbf{p}$, i.e., $\mathbf{p}$ is an eigenvector of A corresponding to $\lambda = 1$ (see Homework 11 to understand what can go wrong in a general case). Let's compute find this eigenvector:

$$(A - I)\mathbf{p} = \mathbf{0} \to \begin{bmatrix} 0 & 0 & 1 & 1/3 \\ 1/2 & 0 & 0 & 1/3 \\ 1/2 & 1/2 & 0 & 1/3 \\ 0 & 1/2 & 0 & 0. \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

After row reducing we obtain the following matrix

$$\begin{bmatrix} -3 & 0 & 0 & 10 \\ 0 & -6 & 0 & 12 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so any multiple of $\begin{bmatrix} 10/3 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ is an eigenvector. We choose the one with entries adding up to one $\mathbf{p} = \frac{1}{28} \begin{bmatrix} 10 \\ 6 \\ 9 \\ 3 \end{bmatrix}$, that is, $p_1 = \frac{10}{28}$, $p_2 = \frac{6}{28}$, $p_3 = \frac{9}{28}$, $p_4 = \frac{3}{28}$.

Problem 8

Read section 10.4 of Strang. What was the most difficult for you?