Name ____

Math 312 - Section 001 - Final Exam (Practice exam)

Wednesday, December 19, 2018, @ 9:00 AM - 11:00 AM

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________________

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8×11 cheat-sheet.

OFFICIAL USE ONLY:

Problem	Points	Your score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Problem 1 [10 points]

Part a. Determine if each of the following systems $A\vec{x} = \vec{b}$ has a solution. If there is a solution, determine whether or not the solution is unique.

a.1) $$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 1 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

a.2) $$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -1 & 3 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}$$

a.3) $$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 4 & -2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

Part b. Find the coefficients for the model below that best fit the data $x = \pi/2, -\pi/2, -\pi/2, \pi/2$, $y = 1, -1, 1 - 1$, $z = 1, 2, 3, 4$ in the least squares sense:

$$z = a \sin(x) + 2^y b + c.$$

Problem 2 [10 points]

Part a. Which of the following matrices are diagonalizable (over the complex numbers)?

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} -13/\pi & 2 \\ 2 & \sqrt{3} \end{bmatrix}$$

Consider the linear differential system

$$\begin{aligned} x' &= x - 3y \\ y' &= -x - y. \end{aligned}$$

Part b. For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?

Part c. Find the exponential matrix e^{At}. Use that the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2$ with corresponding eigenvectors $\vec{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, and $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Part d. Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = 1$ and $y(0) = 0$. What is the ratio $y(t)/x(t)$ as t goes to infinity?

Part e. For what initial conditions $x(0)$, $y(0)$ does the solution $(x(t), y(t))$ to this differential system lie on a single straight line in $\mathbb{R}^2$ for all t? (Hint: You can do this explicitly, as in d), or just thinking on the phase portrait)

Problem 3 [10 points]

Let V be the subspace of vectors in $\mathbb{R}^4$ whose components add up to zero.

Part a. Find a basis for V.

Part b. Apply Gram-Schmidt to find an orthonormal basis for V.

Part c. Find the point on V closest to the point $(1, 0, -1, 1)$.

Problem 4 [10 points]

Let P be the vector space of polynomials of degree less than 2. Let $U = \{1, x, x^2\}$ and $V = \{x, 2 - x, x^2 + 1\}$.

Part a. Find the matrix of the linear transformation T corresponding to the derivative of polynomials, using U as input basis and V as output basis.

Part b. What is the kernel of T?

Part c. What is the image of T?

Problem 5 [10 points]

Given the following basis for the left-nullspace of a matrix A,

$$\{\begin{bmatrix} -16 \\ -1 \\ 3 \end{bmatrix}\},$$

Part a. Find a basis for the column space, $C(A)$.

Part b. Construct a matrix with the above left-nullspace and column space, with the additional condition that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in its nullspace, $N(A)$.

Problem 6 [10 points]

Part a. Suppose you must buy fruit and vegetables (in pounds) for a dinner party. You must buy at least twice as much vegetables as fruit, and you must buy at least 3 and at most 5 pounds of food. And you must have at least 1 pound of vegetables. Suppose one pound of vegetables costs \$4 and a pound of fruit costs \$2. You want to minimize the money spent subject to these constraints.

Write this as an LP in standard form (write it as a maximization problem). Graph the feasible domain and find the solution to the LP using geometric methods.

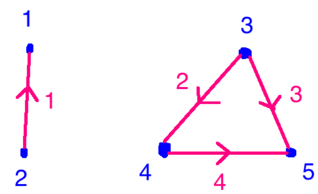
Part b. Given the standard form

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

find the optimal solution using the simplex method.

Problem 7 [10 points]

Part a. Write the incidence matrix A for the following graph, adhering to the numbering given to the edges and vertices.



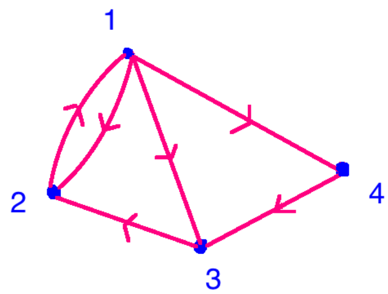
and find a basis for $N(A^T)$ (you may use the graph to find this).

Part b. Given that B is an adjacency matrix for some graph, with

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

find the number of paths of length 2 from node 1 to node 5, and find the degree of node 2, using ONLY the matrix B. Then, draw a graph which has adjacency matrix B.

Part c. Find the Page Rank, with damping $d = 4/5 = .8$ (so $\alpha = 1/5 = .2$) of the following graph:



Problem 8 [10 points]

Suppose that at the end of each year, 80% of the occupants of city A remain in the city, 10% of them move to city B, and 10% of them move to city C; and that 60% of occupants of city B stay in city B, 30% move to city A, and 10% move to city C; and that 70 % of people in city C stay in city C, 20% move to city A, and the rest move to city B.

Part a. If, initially, half of people live in city B and half of people live in city C (with no people living in city A), what is the distribution of people after 1 year?

Part b. What does the population distribution converge to as time goes to infinity?

Part c. How do you know that the limit above exists for any initial distribution?

Problem 9 [10 points]

Part a. Find the SVD of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 4 \end{bmatrix}$$

and use it to an find orthonormal basis for the null space $N(A^T)$; clearly describe which parts of the SVD you are using to deduce this.

Part b. Find a basis for the plane (2d subspace) of best fit through the origin for the points $(1, 1, 1, 1), (1, -1, 2, 1), (1, 0, 3, 0), (-2, 0, 0, 1), (0, 0, 0, -1)$.

You are given part of the (approximate) SVD:

$$\begin{bmatrix} 1 & 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} .6 & -.7 & .4 & .1 \\ .1 & .1 & -.2 & .9 \\ .8 & .4 & .3 & -.2 \\ .1 & .6 & .8 & .1 \end{bmatrix} \Sigma V^T$$

Part c. Write the projection of each of the points onto the plane of best fit in terms of the basis you found above. Use this to plot these projections in a plane. (Hint: you may assume and use that the columns of U are orthonormal).

Part d The following is a singular value decomposition:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}^T = U\Sigma V^T$$

Find the shortest length solution to the least squares problem for

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 5 [10 points]

In each of the following cases, clearly mark the statement as **true** or **false**. Please also **explain your answers** in order to receive credit for this problem.

1. The collection of invertible 4×4 matrices forms a subspace of the collection of 4×4 matrices.

2. If the null-space of a matrix A contains a non-zero vector, then $A\vec{x} = \vec{b}$ has infinitely many solutions for every vector $\vec{b}$.

3. In $\mathbb{R}^3$, the orthogonal complement of a plane is a line.

4. There is at least one value of a for which the matrix $A = \begin{bmatrix} 1 & a \\ 2 & 3 \end{bmatrix}$ has orthogonal eigenvectors.

5. The matrix which projects onto the column space of A is given by $A(A^T A)^{-1}A^T$.

6. If A is a Markov matrix and $\vec{v}$ is a vector, then the entries of $A\vec{v}$ sum up to the same quantity as the entries of $\vec{v}$.

7. If a square matrix A has all its entries strictly negative, then it must have a strictly negative eigenvalue, and one can choose a strictly negative eigenvector corresponding to that eigenvalue.

8. If A is a 2×2 matrix with eigenvalues -1 and 2, and B is a 2×2 matrix with eigenvalues 0 and 2, then $\det((B - I)^2 A^{-1}) = 1$