MIDTERM EXAM 2 (Practice Exam)

MATH 312, GROUP 001

Version A

Name:

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8×11 cheat-sheet.

Problem Number	Possible Points	Points Earned
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1

We measure the vertical position of a point at instants $t = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ and obtain $y = 0.8, 2.1, 0.2$. We think a good model for the movement of the point is $y = a + b \sin t$.

Part a. [5 points] Write down the 3 equations that would be satisfied *if* our model perfectly fitted the data

Part b. [10 points] Find the coefficients a, b that best fit the data.

Part c. [5 points] If we assume the model is right, what are the errors of those three measurements?

Problem 2

[20 points] Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$, and consider the standard basis for $\mathbb{R}^3$: $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Part a. [3 points] What is the dimension of V?

Part b. [5 points] Find an orthonormal basis $\mathcal{V}$ for V.

Part c. [6 points] Let $\mathbf{u} = 3\mathbf{e}_1 + \mathbf{e}_3$, that is, $\mathbf{u}|_{\mathcal{E}} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Find the projection $\mathbf{p}$ of $\mathbf{u}$ onto V. Write this result in the standard basis, $\mathbf{p}|_{\mathcal{E}}$, and in the basis $\mathcal{V}$, $\mathbf{p}|_{\mathcal{V}}$.

Part d. [6 points] The projection of vectors from $\mathbb{R}^3$ onto V is a linear transformation. Find the corresponding matrix when using the basis $\mathcal{E}$ and $\mathcal{V}$.

Problem 3

[20 points] Consider the linear differential system

$$x' = 2x + 3y$$
$$y' = 3x + 2y.$$

Part a. [2 points] For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?

Part b. [6 points] Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.

Part c. [4 points] Find the exponential matrix e^{At}.

Part d. [4 points] Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = 1$ and $y(0) = 3$.

Part e. [4 points] If $x(t)$ and $y(t)$ represent two species that have a mutually symbiotic relationship, say $x(t)$ number of flowers and $y(t)$ number of bees, how many bees per flowers are there in the equilibrium situation (that is, as time goes to infinity)?

Problem 4

[15 points]

Part a. [10 points] Is the following matrix positive semi-definite, i.e., $\mathbf{x}^T M\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$? Prove your answer.

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

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Part b. Given the ellipse $3x^2 + 4xy + 2y^2 = 1$,

b.1 [5 points] Find M such that $\begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} = 1$.

b.2 [5 points] Find the principal axis of the ellipse and the lengths of its semiaxis. Sketch the ellipse.

Problem 5

[20 points: 2.5 points each] In each of the following cases, clearly mark the statement as **true** or **false**. Please also explain your answers in order to receive credit for this problem.

1. Let A be a 2 by 5 matrix such that AA^T has eigenvalues 1 and 2. Then $\dim(N(A^TA)) = 2$.
2. Any matrix with orthonormal columns always has orthonormal rows.
3. The transformation $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(\mathbf{x}) = x_1^2 + x_2$ is linear (here $\mathbf{x} = (x_1, x_2)$).
4. For any M symmetric matrix, it is impossible to find two eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $\mathbf{u}_1^T\mathbf{u}_2 \neq 0$.
5. If A is a 2 by 2 matrix with eigenvalues -1 and 2, and B is a 2 by 2 matrix with eigenvalues 0 and 1, then $\det((B + I)A^{-1}) = 1$.
6. If $A^3 = 0$ for some square matrix A, then all the eigenvalues of A are zero.
7. A symmetric and orthogonal matrix always has eigenvalues equal to 1 or -1.

8. For any square matrices A and B, $\det(A + B) = \det(A) + \det(B)$.