P1

a)
$$\left.\begin{aligned} 0.8 &= a + b\sin(0) \\ 2.1 &= a + b\sin\left(\tfrac{\pi}{2}\right) \\ 0.2 &= a + b\sin\left(\tfrac{3\pi}{2}\right) \end{aligned}\right\} \rightarrow \left\{\begin{aligned} 0.8 &= a \\ 2.1 &= a + b \\ 0.2 &= a - b \end{aligned}\right\}$$

b) Since the above equations do not have a solution, we solve the normal equations to find the least squares solution:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0.8 \\ 2.1 \\ 0.2 \end{bmatrix}}_{\vec{b}} \rightsquigarrow A^T A \vec{x} = A^T \vec{b} \quad \text{// Normal equations.}$$

$$\left.\begin{aligned} A^T A &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \\ A^T \vec{b} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 2.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 3.1 \\ 1.9 \end{bmatrix} \end{aligned}\right\} \rightarrow$$

$$\rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3.1 \\ 1.9 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3.1 \\ 1.9 \end{bmatrix} = \begin{bmatrix} \frac{3.1}{3} \\ 1.9/2 \end{bmatrix}$$

c) If the model is right, we should have measure the following:

$$\left.\begin{aligned} t_1 &= 0 \rightarrow y = a = 3.1/3 \\ t_2 &= \tfrac{\pi}{2} \rightarrow y = a + b = \tfrac{3.1}{3} + \tfrac{1.9}{2} \\ t_3 &= \tfrac{3\pi}{2} \rightarrow y = a - b = \tfrac{3.1}{3} - \tfrac{1.9}{2} \end{aligned}\right\} \begin{aligned} \text{error}_1 &= \left|0.8 - \tfrac{3.1}{3}\right| \simeq |0.8 - 1.0\bar{3}| \simeq 0.2\tilde{3} \\ \text{error}_2 &= \left|\tfrac{3.1}{3} + \tfrac{1.9}{2} - 2.1\right| = \left|\tfrac{6.2 + 5.7 - 12.6}{6}\right| = \tfrac{0.7}{6} \\ \text{error}_3 &= \left|\tfrac{3.1}{3} - \tfrac{1.9}{2} - 0.2\right| = \left|\tfrac{6.2 - 5.7 - 1.2}{6}\right| = \tfrac{0.7}{6}. \end{aligned}$$

$\boxed{P2}$

a) $\dim V = 2$ (plane in $\mathbb{R}^3$)

b) Basis for $V \longrightarrow \left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\} = \{ \vec{a}_1, \vec{a}_2 \}$

Gram-Schmidt:

$\vec{v}_1 = \vec{a}_1$

$$\vec{v}_2 = \vec{a}_2 - \left(\frac{\vec{v}_1 \cdot \vec{a}_2}{\|\vec{v}_1\|^2}\right)\vec{v}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\-1\end{bmatrix} \quad (\vec{v}_2 \cdot \vec{v}_1 = 0 \checkmark).$$

So orthonormal basis is

$$\{\vec{g}_1, \vec{g}_2\} \text{ with } \vec{g}_1 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}, \quad \vec{g}_2 = \frac{1}{\sqrt{1+\frac{1}{2}}}\begin{bmatrix}1/2\\1/2\\-1\end{bmatrix} = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}1/2\\1/2\\-1\end{bmatrix}$$

c)

$$\vec{p} = (\vec{u} \cdot \vec{g}_1)\vec{g}_1 + (\vec{u} \cdot \vec{g}_2)\vec{g}_2 = \frac{3}{\sqrt{2}}\vec{g}_1 + \left(\frac{3\sqrt{2}}{2\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}\right)\vec{g}_2 =$$

$$= \frac{3\sqrt{2}}{2}\vec{g}_1 + \frac{\sqrt{2}}{2\sqrt{3}}\vec{g}_2 \Rightarrow \vec{p}\,|_V = \begin{bmatrix}3\sqrt{2}/2\\ \sqrt{6}/6\end{bmatrix}$$

And

$$\vec{p}\,|_E = \frac{3\sqrt{2}}{2}\underbrace{\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}}_{\vec{g}_1|_E} + \frac{\sqrt{6}}{6}\underbrace{\frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}1/2\\1/2\\-1\end{bmatrix}}_{\vec{g}_2|_E} = \begin{bmatrix}\frac{3}{2}+\frac{1}{6}\\ -\frac{3}{2}+\frac{1}{6}\\ 0-\frac{2}{6}\end{bmatrix} = \begin{bmatrix}5/3\\-4/2\\-1/3\end{bmatrix}$$

d) $M = \left[T(\vec{e}_1)\big|_V \quad T(\vec{e}_2)\big|_V \quad T(\vec{e}_3)\big|_V \right]$

$$T(\vec{e}_1) = (\vec{e}_1\cdot\vec{g}_1)\,\vec{g}_1 + (\vec{e}_1\cdot\vec{g}_2)\,\vec{g}_2 = \frac{1}{\sqrt{2}}\,\vec{g}_1 + \frac{\sqrt{2}}{2\sqrt{3}}\,\vec{g}_2 \rightarrow T(\vec{e}_1)\big|_V = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2\sqrt{3} \end{bmatrix}$$

$$T(\vec{e}_2) = (\vec{e}_2\cdot\vec{g}_1)\,\vec{g}_1 + (\vec{e}_2\cdot\vec{g}_2)\,\vec{g}_2 = \frac{-1}{\sqrt{2}}\,\vec{g}_1 + \frac{\sqrt{2}}{2\sqrt{3}}\,\vec{g}_2 \rightarrow T(\vec{e}_2)\big|_V = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2\sqrt{3} \end{bmatrix}$$

$$T(\vec{e}_3) = \ldots = 0\,\vec{g}_1 - \frac{\sqrt{2}}{\sqrt{3}}\,\vec{g}_2 \rightarrow T(\vec{e}_3)\big|_V = \begin{bmatrix} 0 \\ -\sqrt{2}/\sqrt{3} \end{bmatrix}$$

So

$$M = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{6}/6 & \sqrt{6}/6 & -\sqrt{6}/3 \end{bmatrix}$$

Check: $M\,\vec{u}\big|_E = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{6}/6 & \sqrt{6}/6 & -\sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ \sqrt{6}/6 \end{bmatrix} = \vec{u}\big|_V$

$\boxed{P3}$

a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

b) $A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} \rightarrow (2-\lambda)^2 - 9 = 0 \Leftrightarrow$

$\Leftrightarrow 2 - \lambda = \pm 3 \Leftrightarrow \lambda = 2 \mp 3$ $\nearrow \lambda_1 = 5$ $\searrow \lambda_2 = -1$.

$A + I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

c) $e^{At} = S e^{Dt} S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \frac{-1}{2} =$

$= \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{2} = \frac{1}{2} \begin{bmatrix} e^{5t} + e^{-t} & e^{5t} - e^{-t} \\ e^{5t} - e^{-t} & e^{5t} + e^{-t} \end{bmatrix}$

d) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} =$

$= \frac{1}{2} \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow$ $x(t) = 2e^{5t} - e^{-t}$, $y(t) = 2e^{5t} + e^{-t}$ //.

e) $\lim\limits_{t\to+\infty} \frac{y(t)}{x(t)} = \lim\limits_{t\to\infty} \frac{2e^{5t}+\cancel{e^{t}}}{2e^{5t}-\cancel{e^{t}}} = 1$ //

$\boxed{P4}$

a) $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \longrightarrow (1-\lambda)^2 - 4 = 0 \iff 1-\lambda = \pm 2 \iff \lambda = 1 \mp 2$ $\nearrow \lambda_1 = -1$ $\searrow \lambda_2 = 3$.

↳ So M is not positive semi-defite (it has a negative eigenvalue).

Proof:

Option 1): Prove that $\vec{x}^T M \vec{x} \geq 0 \Rightarrow$ All $\lambda \geq 0$.

↳ Let $\lambda, \vec{u}$ eig. Since M is symmetric, both λ and $\vec{u}$ are real.
Now,

$$M\vec{u} = \lambda\vec{u} \longrightarrow \underbrace{\lambda \vec{u}^T\vec{u}}_{=\lambda\|\vec{u}\|^2} = \underbrace{\vec{u}^T M \vec{u}}_{\geq 0} \Rightarrow \lambda\|\vec{u}\|^2 \geq 0 \Rightarrow \lambda \geq 0.$$

$\uparrow$ $\vec{u}$ eigvetor ($\vec{u} \neq 0$) so $\|\vec{u}\| > 0$

Option 2): Find one $\vec{x}$ such that $\vec{x}^T M \vec{x} < 0$:

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 4 + 1 = -2 \text{ //}.$$

(eigenvector corresponding to $\lambda_1 = -1$)

b) In lecture notes of review session.

$\boxed{P5}$

1) False.

A 2×5 $\Rightarrow$ AA^T is 2×2 , A^TA is 5×5 $\Big\{$ They share nonzero eigenvalues, therefore,

A^TA has eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \lambda_4 = \lambda_5 = 0 \Rightarrow$

$\Rightarrow \dim N(A^TA) = 3$

A^TA is symmetric, so spectral theorem applies (we can always diagonalize).

2) False: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ (It is true <u>if</u> the matrix is square).

3) False: $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1$

$T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = 1 \neq -T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = -1$

4) False: If an eigenvalue is repeated twice, its eigenspace (that is, the nullspace of $A-\lambda I$) has dimension 2, so we can take any 2 l.i. vectors (not necessarily orthogonal).

Ex: $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \lambda_1 = \lambda_2 = 1$

All $\vec{x} \in \mathbb{R}^2 \setminus \{\vec{0}\}$ are eigenvectors!

5) False:

$\det((B+I)A^{-1}) = |B+I|\,|A^{-1}| = |B+I|\frac{1}{|A|}$ where

$\det(B+I) = \text{product of eigenvalues of } B+I = 1\cdot 2 = 2$ (equal eigenvalues of B plus 1).

$|A| = -1\cdot 2 = -2$

$\Rightarrow |(B+I)A^{-1}| = \frac{2}{-2} = -1.$

6) True: $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x} \Rightarrow \underbrace{A^3\vec{x}}_{=\vec{0}} = \lambda^2 A\vec{x} = \lambda^3\vec{x}$

$\Rightarrow \lambda^3\vec{x} = \vec{0} \Rightarrow \lambda = 0$ $(\vec{x} \neq \vec{0})$

7) True: Symmetric $\Rightarrow$ real eigenvalues (spectral theorem).

Q orthogonal $\Rightarrow$ $Q\vec{x} = \lambda\vec{x} \Rightarrow \|Q\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\|\vec{x}\| \Rightarrow$

$\|Q\vec{x}\| = \|\vec{x}\|$ $\Rightarrow |\lambda| = 1 \Rightarrow \lambda = \pm 1$.

λ real.

8) False: $\det(I + I) = \det(2I) = 2^n \det(I) = 2^n$.

$\det(I) + \det(I) = 2 \neq$.