Extra Problems for Midterm 2

Problem 1

Suppose you know that the eigenvalues of a 3×3 matrix A are $0, 1, 2$ with corresponding eigenvectors $\vec{v}_0, \vec{v}_1$, and $\vec{v}_2$.

1. What are the eigenvectors and eigenvalues of A^2?
2. What are the eigenvectors and eigenvalues of $A + Id$?
3. Is A invertible? If so what are the eigenvectors and eigenvalues of A^{-1}?

Problem 2

We want to find the curve $y = a + 2^t b$ that gives the best fit (in the least squares sense) to the data $t = 0, 1, 2$, $y = 6, 4, 0$.

1. Write down the 3 equations that would be satisfied if the curve went though all 3 points.
2. Find the coefficients a, b of the curve of best fit $y = a + 2^t b$.

Problem 3

A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

1. Apply the Gram-Schmidt process to find two orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$ which also span V.
2. Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.
3. Find the best possible (i.e., least squared error) solution to the linear system

$$Q \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Problem 4

Consider the planes $H = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and $V = \{(x, y, z) \in \mathbb{R}^3 : z = y\}$. We will use the following basis for H:

$$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}.$$

1. Write the vector $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ in the basis $\mathcal{E}$. That is, give the coordinates of $\mathbf{u}$ in $\mathcal{E}$.
2. Find an orthonormal basis $\mathcal{V}$ for V.
3. Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis $\mathcal{V}$.
4. The projection of vectors in V onto H is a linear transformation. Find the matrix of this linear transformation using the basis $\mathcal{V}$ and $\mathcal{E}$.
5. Find the area of the triangle with vertex $(0, 0, 0)$, $(1, 1, 1)$ and $(0, 1, 1)$.

Problem 5

Consider the following basis of $\mathbb{R}^3$: $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \}$, where the coordinates of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are given with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

1. Given a general vector $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ written in the standard basis (i.e., $\mathbf{u} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3$), find its coordinates in $\mathcal{B}$

Problem 6

The polynomials $\vec{u}_1 = 1$, $\vec{u}_2 = x - 2$, and $\vec{u}_3 = (x - 2)^2$ form a basis for the space of (at most) quadratic polynomials in x, as do the polynomials $\vec{v}_1 = 1$, $\vec{v}_2 = x + 1$, and $\vec{v}_3 = (x + 1)^2$. Find the change of basis matrix from $\{u_i\}$ to $\{v_i\}$ and use it to find numbers a, b, c such that $-1(x - 2) + 3(x - 2)^2 = a + b(x + 1) + c(x + 1)^2$.

Problem 7

Find the limit of A^k as k goes to infinity for

$$A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}$$

Problem 8

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & b \end{bmatrix}$$

For which values of b does A have distinct eigenvalues?

Problem 9

Let P be a matrix that projects vectors of $\mathbb{R}^3$ onto the plane $z = 0$. What are the eigenvalues and eigenvectors of P?

Problem 10

Consider the linear differential system

$$\begin{aligned} x' &= x + 3y \\ y' &= 2x + 2y. \end{aligned}$$

1. For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?
2. Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.
3. Write the exponential matrix e^{At}
4. Find the solution $x(t)$ and $y(t)$ to this linear differential system subject to the initial conditions $x(0) = -5$ and $y(0) = 5$.
5. If $x(t)$ and $y(t)$ represent two species that have a mutually symbiotic relationship, say $x(t)$ number of flowers and $y(t)$ number of bees, how many bees per flowers are there in the equilibrium situation (that is, as time goes to infinity)?

Problem 11

Find an orthogonal matrix Q and a diagonal matrix D such that $M = QDQ^T$, where

$$M = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Is M positive definite?

Problem 12

Given the ellipse $3x^2 + 4xy + 2y^2 = 1$,

1. Find M such that $\begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} = 1$.

2. Find the principal axis of the ellipse and the lengths of its semiaxis. Sketch the ellipse.

Problem 13

Which of the following matrices are positive semi-definite?

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

True or False

1. If A is a 3x3 matrix with determinant 1, then $2A$ has determinant 6.

2. If A is a square matrix and B is obtained from A via row operation R2'=R2+3R1, then B has the same eigenvalues as A.

3. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ corresponding to distinct eigenvalues, then $\mathbf{u}_1^T \mathbf{u}_2 = 0$.

4. A definite positive matrix always has an inverse.

5. If A is invertible and has one eigenvalue λ, then $1/\lambda$ is an eigenvalue of A^{-1}.

6. Is the following transformation linear? $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(\mathbf{x}_0) =$ solution to the system given by $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \mathbf{x}$ with initial condition $\mathbf{x}_0$.

7. Is the following transformation linear? $\begin{aligned} T &: \mathbb{R}^3 \to \mathbb{R}^3 \\ \mathbf{x} &\to T(\mathbf{x}) = \min_{i=1,2,3} x_i \end{aligned}$.

8. If Q is an orthogonal matrix, then the corresponding linear transformation preserves lenghts and angles, i.e., length of $Q\mathbf{x}$ is equal to length of $\mathbf{x}$ and the angle between $\mathbf{x}$ and $\mathbf{y}$ is equal to the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$.

9. An square matrix with orthonormal columns always has orthonormal rows.

10. A 3 by 3 symmetric matrix with eigenvalues $0, 0, 1$ always has rank(A)=1.

11. A 2 by 2 matrix that rotates every vector 90^{o} cannot have any real eigenvalues.

12. The matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$ has an eigenvalue equal to 9.

13. The matrix $A^T A$ is always positive semidefinite.

14. A basis for eigenvectors for nonzero eigenvalues of A is a basis for $C(A)$ for any matrix A.

15. The eigenvectors for the zero eigenvalue are the null space of A.

16. The only upper triangular 3×3 matrix with 1s on the diagonal which is diagonalizable is the identity matrix.