<u>Solutions</u> to previous Midterm exam

$\boxed{P1}$

a) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -6 & 0 \\ -3 & -10 & 0 \end{bmatrix}$

b) They are "written" in L:

1) Subtract (-2) times row 1 from row 2 ($l_{21} = -2$) ($\equiv$ add 2 times row 1 from row 2)

2) Subtract (-3) times row 1 from row 3 ($l_{31} = -3$) ($\equiv$ add 3 times row 1 from row 3)

3) Subtract 1 times row 2 from row 3 ($l_{32} = 1$)

c) Let's call $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Then $Ax = b \rightsquigarrow LUx = b$. Denote $y = Ux$. Thus, we can solve:

1) $Ly = b$ to find y (forwards ~~elimination~~ substitution)

2) With y known, solve $Ux = y$ for x (backwards substitution).

Of course, they need to be solved in order (we need to know y to solve $Ux = y$ for x).

d) $[U \; I] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$

$U^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}$ ((Check it!))

e) A is invertible since it has three pivots (U is invertible!).

$A = LU \to A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$ and we know that $L^{-1}A = U$,

with

$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

((Check it!))

So,

$A^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 3 \\ 0 & 3/2 & -1 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$ ((Check it!))

$\boxed{P2}$

a) We know that basis for $C(B^T)$ is the same as a basis for $C(R^T)$:

$$\left\{ \begin{bmatrix}1\\0\\3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \right\}$$

b) It is formed by the columns of B where pivots will appear, i.e.,

$\left\{ \begin{bmatrix} b_1 \end{bmatrix}, \begin{bmatrix} b_2 \end{bmatrix} \right\}$ where b_1, b_2 are the columns 1 and 2 of B.

$$B = \Pi R = \begin{bmatrix}2&2&0\\0&2&2\\2&0&2\end{bmatrix}\begin{bmatrix}1&0&3&2\\0&1&1&0\\0&0&0&0\end{bmatrix} = \begin{bmatrix}2&2&8&4\\0&2&2&0\\2&0&0&4\end{bmatrix},$$

so basis for $C(B) = \left\{ \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix} \right\}$.

<u>Remark</u>: We only needed to compute $b_1 = Rr_1$, $b_2 = Rr_2$:

$$\begin{bmatrix}2&2&0\\0&2&2\\2&0&2\end{bmatrix}\begin{array}{c}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\\ \;\; r_1 \;\; r_2\end{array} = \begin{bmatrix}2&2\\0&2\\2&0\end{bmatrix}$$

c) $N(B) = N(R) \rightarrow$

basis of $N(B) \equiv$ basis of $N(R)$, $R = \begin{bmatrix}\boxed{1}&0&3&2\\0&\boxed{1}&1&0\\0&0&0&0\end{bmatrix}$ $\begin{array}{l} x_3 = \alpha, \\ x_4 = \beta \end{array}$

free

$x_1 = -3\alpha - 2\beta$
$x_2 = -\alpha$

$$\left.\begin{matrix} x_1 = -3\alpha - 2\beta \\ x_2 = -\alpha \end{matrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis $N(B) = \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

d)

$B = MR \Rightarrow M^{-1}B = R$ and we look at the last row of R (free):

$$\underbrace{\begin{bmatrix} -1/4 & 1/4 & 1/4 \end{bmatrix}}_{\in N(B^T)} B = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

And we know that $\dim N(B^T) + \underbrace{\dim C(B^T)}_{=2} = m = 3 \Rightarrow$

$\Rightarrow \dim N(B^T) = 1$.

So, basis for $N(B^T) = \left\{ \begin{bmatrix} -1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \right\}$.

e) Give A an $m \times n$ matrix, it holds that:

$$\left.\begin{matrix} \bullet\ C(A) \text{ subspace in } \mathbb{R}^m \\ \bullet\ C(A^T) \text{ ,, in } \mathbb{R}^n \\ \bullet\ N(A) \text{ ,, in } \mathbb{R}^n \\ \bullet\ N(A^T) \text{ ,, in } \mathbb{R}^m \end{matrix}\right\} \left.\begin{matrix} \dim C(A) = \dim C(A^T) = \text{rank} \equiv r \\ \dim N(A) = n - \dim C(A) = n - r \\ \dim N(A^T) = m - \dim C(A^T) = m - r \end{matrix}\right\}$$

Moreover, $C(A)^\perp = N(A^T)$, $C(A^T)^\perp = N(A)$

For our matrix B:

$$\left.\begin{aligned} \dim C(B) &= 2 = \dim C(B^T) \\ \dim N(B) &= 2 = 4-2 \\ \dim N(B^T) &= 1 = 3-2 \end{aligned}\right\}$$

- $\underline{C(B)^{\perp} = N(B^T)}$:

basis of $C(B) = \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$, so $C(B)^{\perp}$ is given by all vectors in $\mathbb{R}^3$ such that

$$\left.\begin{aligned} 2x_1 + 2x_3 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A basis for $C(B)^{\perp}$ is thus $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$.

Notice that $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = -4 \underbrace{\begin{bmatrix} -1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}_{\text{Basis of } N(B^T)}$ ✓.

- $\underline{C(B^T)^{\perp} = N(B)}$

basis of $C(B^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow (C(B^T))^{\perp}$ is given by $\vec{x} \in \mathbb{R}^4$:

$$\left.\begin{aligned} x_1 + 3x_3 + 2x_4 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\} \begin{bmatrix} \boxed{1} & 0 & 3 & 2 \\ 0 & \boxed{1} & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\text{Basis for } C(B^T)^\perp = \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Basis of } N(B)$$

$\boxed{P3}$

a) $Bx = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}}_{=v}$, $B = MR \rightarrow Bx = v \Leftrightarrow MRx = v \Leftrightarrow$

$$\Leftrightarrow Rx = M^{-1}v = \begin{bmatrix} 1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \frac{2}{4} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

So, we want to solve

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 2 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = \alpha \\ x_4 = \beta \end{array} \text{(free)},$$

$$\left.\begin{array}{l} x_1 = 0 - 3x_3 - 2x_4 = -3\alpha - 2\beta \\ x_2 = 1 - x_3 = 1 - \alpha \end{array}\right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

b) $\begin{bmatrix} \boxed{1} & 0 & 3 & 2 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The nullspace is the same, so:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

P4

a), b) c), d), e), g) done in class (quizz).

f) True: indeed, if A, B, C are invertible, then

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$.

<u>Remark</u>: But, if ABC is invertible it is <u>NOT</u> true (in general) that A, B and C are invertible.

↳ Think of A^TA: this is usually invertible, but A and A^T can be rectangular!