# Extra Problems for Midterm 1

## Problem 1

1. What 3 by 3 matrix will add row 2 to row 3?
2. What matrix adds row 3 to row 2 and at the same time row 2 to row 3?
3. What matrix adds row 1 to row 3 and then adds row 3 to row 1?

Which of the matrices above are not invertible? Explain. Find the inverses when they exist.

## Problem 2

Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a row reduced $R\mathbf{x} = \mathbf{b}$: the complete solution is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

1. What is the 3 by 3 reduced row echelon matrix $R$ and what is $\mathbf{d}$?
2. If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects $R$ and $\mathbf{d}$ to the original $A$ and $\mathbf{b}$? Use this matrix to find $A$ and $\mathbf{b}$.

## Problem 3

Suppose $A$ is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

1. Find a set of independent equations describing $N(A)$ and a basis for $N(A)$.
2. Find a basis and a set of independent equations for $C(A)$.

## Problem 4

1. When a set of vectors $\mathbf{u_1}, \ldots, \mathbf{u_k}$ are linearly independent? Give the definition.
2. Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}$. Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of $A$? Explain your answer.

### Problem 5

Let $A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}$.

1. By elimination find the pivot columns of $A$ and determine the rank of $A$.
2. Find the complete solution to $Ax = 0$
3. For which number $\alpha$ does $A\mathbf{x} = \begin{bmatrix} 3 \\ 9 \\ \alpha \end{bmatrix}$ have a solution?

### Problem 6

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & \alpha \end{bmatrix}$.

1. If the columns of $B$ are vectors in the nullspace of $A$, find the product matrix $AB$.
2. For which values of $c$ is the matrix $A$ invertible?
3. Find the $A = LU$ (lower triangular times upper triangular) decomposition of $A$ when $c = 0$.

### Problem 7

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}$.

1. Which are the pivot columns and which are the free columns of $A$ and $M$?
2. For which $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$ are there solutions to $A\mathbf{x} = \mathbf{b}$? Write down the complete solution for those $\mathbf{b}$ (you will need to compute $N(A)$).
3. Explain why $M$ has an inverse (without using determinants). Find it using Gauss-Jordan

### Problem 8

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}$. Write it as the product of elementary matrices. Find the inverses of these elementary matrices and write $A^{-1}$ as a product of them.

## Problem 9

For a 3 by 3 matrix $A$, suppose all three multipliers are $l_{21} = l_{31} = l_{32} = 3$. Each $l_{ij}$ multiplies pivot row $j$ when it is subtracted from row $i$.

1. Assuming no row exchanges, what is $A$, if elimination reaches $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & g \end{bmatrix}$?
2. For $g = 0$, find a basis for $N(A)$.
3. If $g \neq 0$, find a basis for $N(U)$ and a basis for $N(A)$.

## Problem 10

$A$ is a 2 by 4 matrix with nullspace given by all vectors of the form

$$\alpha \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \alpha, \beta \in \mathbb{R}$$

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1. Find the reduced row echelon form $R$ of $A$.
2. What is the column space of $A$?
3. What is the complete solution to $R\mathbf{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$?
4. Show that columns, 2,3,4 of A are linearly dependent by finding a linear combination of them that gives the zero vector.

## Problem 11

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{bmatrix}$.

1. For $k = 10$, find the $A = LU$ factorization of $A$.
2. For $k = 10$, solve the system $A\mathbf{x} = \begin{bmatrix} 3 & 10 & 20 \end{bmatrix}^T$.
3. For which values of $k$ is $A$ singular (i.e., the inverse does not exists)?
4. Find all values of $k$ for which the system $A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ has infinitely many solutions. (You don't need to solve the system in this part)
5. Find all values of $k$ for which the system $A\mathbf{x} = \begin{bmatrix} 10 & 1 & 2018 \end{bmatrix}^T$ has exactly one solution. (You don't need to solve the system in this part).

## Problem 12

Let $k \in \mathbb{R}$. Consider the following system

$$\begin{aligned} kx + 3y &= 6, \\ 3x + ky &= -6 \end{aligned}.$$

Which values of $k$ make the system solvable? Find the complete solution in those cases.

## Problem 13

Which of the following sets are vector subspaces of $\mathbb{R}^3$? Explain your answer.

1. All vectors $(x, y, z)$ such that $10x + y + 2018z = 0$.
2. All vectors $(x, y, z)$ such that $x + y + z \leq 0$
3. All vectors $(x, y, z)$ such that $x + y + z = 0$ **and** $x + 2y + 3z = 0$.
4. All vectors $(x, y, z)$ such that $x + y + z = 0$ **or** $x + 2y + 3z = 0$.
5. All vectors $\begin{bmatrix} x & y & z \end{bmatrix}^T$ such that the system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ has a solution.

## Problem 14

1. Find a 3 by 3 matrix $A$ whose column space is the plane $x + y + z = 0$ in $\mathbb{R}^3$.
2. Explain why that 3 by 3 matrix cannot be invertible.
3. Does there exist a matrix $B$ whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$ and whose nullspace is spanned by $(1, 2, 3, 6)$. If so, construct such a $B$. If not, explain why not.
4. Explain if the following set of matrices is a vector space or not: All 3 by 3 matrices with the column vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ in their column space.