Programa de Doctorado Matemáticas

PhD Dissertation

Global regularity for incompressible fluid interfaces

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Presentación

Esta memoria está dedicada al estudio de tres problemas de frontera libre dadas por interfases entre fluidos incompresibles: parche de temperatura en Boussinesq, parche de densidad en Navier-Stokes y el problema de Muskat. Estos problemas proceden de diferentes sistemas físicos cuya evolución puede describirse mediante ecuaciones en derivadas parciales parabólicas no lineales y no locales. El trabajo se centra en propiedades cualitativas de las soluciones, tales como *ill-posedness* o regularidad de la interfase para todo tiempo.

Las ecuaciones de Boussinesq son ampliamente usadas como una adecuada aproximación del movimiento de fluidos en fenómenos de convección natural. En estos procesos, el movimiento fluido se debe a la acción de la gravedad sobre variaciones de densidad inducidas por cambios de temperatura, sin agentes externos del movimiento. La aproximación de Boussinesq resalta este hecho al asumir constante la densidad en todos los términos salvo el de gravedad.

Desde el punto de vista matemático, el mayor interés radica en la conexión entre el sistema de Boussinesq bidimensional y las ecuaciones de Euler y Navier-Stokes en tres dimensiones. En contraste con estas últimas en el plano, donde la ecuación de la vorticidad no presenta término cuadrático, Boussinesq 2d consigue capturar el fenómeno de *vortex stretching.* Al igual que en las ecuaciones de Euler y Navier-Stokes en 3d, la regularidad para todo tiempo de las ecuaciones de Boussinesq en dos dimensiones sin viscosidad ni difusividad térmica sigue siendo un destacable problema abierto. De hecho, en ese caso, las ecuaciones de Boussinesq pueden identificarse formalmente con las ecuaciones de Euler 3d en el caso axisimétrico con rotación, lejos del eje. Si se consideran fluidos viscosos o difusividad térmica no nula, entonces en 2d no pueden producirse singularidades en tiempo finito partiendo de soluciones con energía finita.

En el segundo capítulo, se consideran las ecuaciones de Boussinesq, con viscosidad pero sin difusión, para dato inicial de tipo *parche.* Es decir, la temperatura inicial está dada por la función característica de un dominio acotado. Desde el remarcable resultado de regularidad global para el parche de vorticidad en Euler 2d, se han producido numerosos trabajos sobre soluciones tipo parche en distintos modelos. En particular, se ha encontrado evidencia numérica de colapso puntual al mismo tiempo que la curvatura se hace infinita en el sistema surface-quasi-geostrófico. En ambos problemas, una cantidad escalar, la temparatura, es transportada por un campo de velocidad definido a su vez a través de aquélla. No obstante, se mostrará que la curvatura de un parche de temperatura en Boussinesq no puede hacerse infinita en tiempo finito. Además, se probarán resultados análogos mostrando que la frontera de los parches mantiene su regularidad inicial, medida en espacios de Hölder, para todo tiempo. Para los casos de baja regularidad, se hará uso de la estructura parabólica del modelo aplicando resultados de máxima regularidad para la ecuación del calor. A partir de éstos, una nueva cancelación, encontrada en los operadores integrales singulares de tipo parabólico dados por las segundas derivadas de los núcleos del calor y su combinación con transformadas de Riesz, permitirá lograr el control de la curvatura. Avanzando aun más, se aprovechará la regularidad adicional de la velocidad en la dirección tangencial al parche para propagar interfases más regulares. Estos resultados se encuentran publicados en [62].

Las ecuaciones de Navier-Stokes no homogéneas modelan flujos incompresibles de fluidos con densidad variable. Son ampliamente usadas, por ejemplo, en dinámica oceánica. Sirven también para describir un sistema de dos o más líquidos inmiscibles. Matemáticamente, la teoría de soluciones fuertes para Navier-Stokes inhomogéneo está aun incompleta incluso para el caso bidimensional, mientras que en tres dimesiones abarcan el conocido problema del Milenio.

Anque la existencia globalmente en tiempo de soluciones débiles con energía finita se conoce desde hace tiempo, hasta hace muy poco la teoría de soluciones fuertes requería, o bien densidad inicial positiva y al menos continua, o bien densidad inicial regular. Únicamente en los últimos años estas restricciones se han superado parcialmente. En su libro de 1996, P.-L. Lions propuso el llamado *problema del parche de densidad*: suponiendo que la densidad inicial está dada por un parche, la pregunta es si éste se propaga con la velocidad, manteniendo su frontera la misma regularidad que la interfase inicial. La teoría de soluciones renormalizadas de Di Perna y Lions garantiza que la evolución del parche conserva el volumen, pero no aporta información sobre la regularidad de la frontera.

En el capítulo tercero, se da una respuesta positiva para el caso en el que la interfase inicial entre los líquidos tiene vector tangente bien definido y con regularidad Hölder. Se admite cualquier salto de densidad y cualquier tamaño de la velocidad inicial. Además, la estrategia de la prueba permite tratar el caso límite de dos derivadas, dando así control sobre la curvatura del parche. El parche evoluciona de acuerdo a una ecuación de transporte, dada por la conservación de la masa. Para propagar regularidad del mismo, hace falta primero obtener una ganancia de regularidad para la velocidad. Como la densidad está dada por una función salto, y por tanto de baja regularidad, el acoplamiento quasilineal entre la densidad y la velocidad hace que la ganancia parabólica de regularidad sea difícil de conseguir por métodos estándares. Así, el uso de estimaciones de energía con pesos en tiempo y la mayor regularidad de la derivada convectiva son cruciales en ese paso. Combinando diferentes técnicas, es posible construir la prueba a pasos, usando los resultados para interfases poco regulares en los de más alta regularidad. Los resultados han sido publicados en [64].

El movimiento de dos fluidos incompresibles e inmiscibles en un medio poroso da lugar a un importante problema de frontera libre conocido como *problema de Muskat*. Muskat lo planteó en primer lugar, basándose en la ley experimental de Darcy, para modelar el comportamiento del agua a través de suelos con petróleo en los procesos de bombeo de las industrias petrolíferas. En la aproximación de Darcy, la velocidad, en lugar de la aceleración, es proporcional al gradiente de presiones más las fuerzas externas, tales como la gravedad.

Considerando propiedades constantes pero distintas para cada fluido, las ecuaciones de Darcy se reducen a una ecuación que describe la evolución de la interfase fluida.

En el último capítulo, se estudia la existencia, unicidad y regularidad globalmente en tiempo de soluciones en espacios críticos en el régimen estable, es decir, cuando el fluido más denso se encuentra debajo. Se hará considerando tanto densidades como viscosidades distintas para cada fluido, y para pendientes iniciales en la interfase no necesariamente pequeñas, sino simplemente acotadas por una constante explícita. Además, se muestra que la interfase se vuelve instantáneamente analítica y se aplana con el tiempo, dando tasas óptimas del decaimiento medido en distintas normas. Finalmente, se verá que el caso inestable está mal propuesto incluso considerando soluciones de muy baja regularidad. Estos resultados pueden consultarse en [63].

Abstract

In this thesis, three incompressible fluid interface problems are studied: Boussinesq temperature patch, Navier-Stokes density patch and the Muskat problem. They come from different physical scenarios whose evolution can be described by nonlinear and nonlocal parabolic partial differential equations. The work focuses on qualitative properties of the solutions, such as ill-posedness or global-in-time regularity of the interface.

Boussinesq equations are widely used as an accurate approximation of the full density-dependent fluid equations to model phenomena dominated by natural convection. In natural convection, fluid motion is due to gravity acting on density variations induced by temperature gradients without external sources. The Boussinesq approximation emphasizes this fact by assuming constant density in all terms except the buoyancy.

From the mathematical point of view, the main interest lies on the connection between the two-dimensional Boussinesq system and the three-dimensional Navier-Stokes and Euler equations. In contrast with Navier-Stokes on the plane, where the vorticity equation does not have a quadratic term, 2d Boussinesq still captures the phenomenon of vortex stretching. As in 3d Euler and Navier-Stokes equations, global well-posedness of the 2d inviscid and non-diffusive Boussinesq system remains an outstanding open problem. Indeed, in this setting, the 2d Boussinesq equations can be formally identified with the equations for 3d Euler asixymmetric swirling flows away from the axis of singularity. When viscous fluids or thermal diffusivity are considered, global well-posedness in 2d is known for finite-energy initial data.

In the second chapter, we consider the viscous Boussinesq equations without diffusion for singular initial data of *patch* type. That is, the initial temperature is given by the characteristic function of a bounded domain. Since the non-expected global regularity results for the vortex patch problem in 2d Euler, there has been a long tradition for the patch problem in different models. In particular, numerical evidence of pointwise collapse with curvature blow-up was found in the study of patch solutions for the Surface-Quasi-Geostrophic active scalar system. When diffusion is neglected, the Boussinesq approximation also becomes an active scalar system, where the relation between the velocity field and the scalar is noticeably nonlinear. Nevertheless, we show that the curvature of the boundary of a patch cannot blow up in finite time. We also give analogous global-in-time results for lower and higher regularity interfaces measured in Hölder spaces. For the lower regularity regime, we make use of the parabolic structure of the model by applying maximal regularity results for the heat equation. Building upon this result, a new cancellation found in the singular integral parabolic operators given by two spatial derivatives of the heat kernels and its combination

with Riesz transforms allows us to gain control of the curvature. Going one step further, we take advantage of the extra regularity of the velocity in the tangential direction to propagate more regular interfaces. These results have been published in [62].

The inhomogeneous Navier-Stokes equations model physical systems in which fluids of non-constant densities move as an incompressible flow. This regime is the case in many geophysical problems, such as oceanic fluid dynamics. More prominently, these equations can be used to describe a system of two or more immiscible liquids. Mathematically, the theory of strong solutions for inhomogeneous Navier-Stokes is still not complete even in the 2d case, while the 3d case encompasses the well-known Navier-Stokes Millennium problem.

Although global weak solutions with finite energy were established long ago, until very recently the strong solutions theory required the density to be either positive and at least continuous, or smooth. Only in the last few years these constraints have been partially overcome. In his book of 1996, P.-L. Lions proposed the so-called *density patch problem*: assuming that the initial density is given by a patch, the question is whether the flow propagates the patch with the same regularity as the initial interface. The renormalized solutions theory of Di Perna and Lions guarantees that the evolution of the patch preserves the volume, but nothing can be said about the regularity of the boundary from this weak solution framework.

In the third chapter, we give a positive answer when the initial interface between the two liquids has $C^{1+\gamma}$ regularity or greater, for any density jump and any size of the initial velocity. More remarkably, our proof also works in the limit case of $W^{2,\infty}$, thus providing control of the curvature. In this problem, the active scalar equation is given by the mass conservation principle and the relationship with the velocity by the inhomogeneous Navier-Stokes momentum equations. To propagate regularity of the scalar, a gain of regularity for the velocity is first needed. As the density is given by a patch, and hence is of low regularity, the quasilinear coupling between the density and the velocity makes the parabolic gain of regularity difficult to achieve by standard procedures. Thus, the use of time-weighted energy estimates and the higher regularity of the convective derivative are crucial in this step. By a combination of different techniques, we can bootstrap our result from low regular initial interfaces to higher ones, thus providing a unified approach. These results have been published in [64].

The motion of two incompressible immiscible fluids through a porous medium gives rise to an important free boundary problem called the Muskat problem. Muskat first derived it based on the experimental Darcy's law to model the behavior of water through oil sand in typical pumping processes in the petroleum industry. In Darcy's approximation, the velocity, instead of the acceleration, is proportional to the gradient of the pressure plus the external forces such as gravity. Assuming constant but different properties for each fluid, Darcy's equations are reduced to one describing the evolution of the fluid interface.

In the last chapter, we study global-in-time existence, uniqueness and regularity of solutions in critical spaces in the stable regime, i.e., when the more dense fluid is at the bottom. We do it for the actual physical case where the fluids have both different densities and viscosities, and for initial slopes not necessarily small but just bounded by an explicit constant. Moreover, we improved the previous known constants for the non-viscosity jump case. In addition, we show that the interface becomes instantly analytic and flattens in time, giving large-time decay rates for the solution. We also show ill-posedness in unstable situations even for low regularity solutions. These results can be found in [63].

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Contents

Chapter 1

Preliminaries

In this chapter, we briefly describe the physical scenarios that give rise to the fluid interface problems studied in the following chapters. We also include the equations that will be used and comment some of their basic characteristics.

1.1 Boussinesq approximation

In natural convection phenomena fluid flow generates due to the effect of buoyancy forces. Temperature gradients induce density variations from an equilibrium state, which gravity tends to restore. These flows are usually characterized by small deviations of the density with respect to a stratified reference state in hydrostatic balance. Potential energy is thus the main agent of movement, compared to inertia. Oberbeck was the first to notice by linearization that the buoyancy effect was proportional to temperature deviations [98], and later Boussinesq [12] completed the model based on physical assumptions. It has been since then one of the main ingredients in geophysical models, from ocean and atmosphere dynamics to mantle and solar inner convection, as well as a basic tool in building environmental engineering.

In dimensionless variables, the Boussinesq equations in $\mathbb{R}^d$ are given by the following expression

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u = -\nabla P + \dfrac{1}{\mathrm{Re}} \Delta u + \theta e_d, \\ \theta_t + u \cdot \nabla \theta = \dfrac{1}{\mathrm{Pe}} \Delta \theta, \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}^d$ denotes the velocity, $P : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ represents the pressure deviation from the hydrostatic one and $\theta : \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ symbolizes the temperature variations. The symbol $\Delta = \sum_{i=1}^d \partial_i^2$ denotes the Laplacian and $\nabla = (\partial_{x_i}, \ldots, \partial_{x_d})$ denotes the gradient, $x = (x_1, \ldots, x_d)$ is the space variable and $t \geq 0$ the time. The Reynolds number, Re, indicates the ratio of fluid inertial and viscous forces while the Péclet number, Pe, compares the rates of advective and diffusive heat transport. They are thus inversely proportional to the viscosity ν and thermal diffusivity α constants, respectively.

Roughly, to obtain this system from the equations for density-dependent fluids, one first substitutes

$$\rho(x,t) = \rho_m + \tilde{\rho}(x,t),$$

considering that the density variations, $\tilde{\rho}$, are small compared to the background state, ρ_m, i.e., $\tilde{\rho}/\rho_m \ll 1$. In addition, the density is assumed to be linear with respect to the temperature

$$\tilde{\rho} = -\rho_m \beta(\Theta - \Theta_m),$$

where β is the thermal expansion coefficient and the real temperature Θ is related to θ by the Richardson number, Ri, as follows

$$\theta = -\mathrm{Ri}\frac{\tilde{\rho}}{\rho_m} = \mathrm{Ri}\ \beta(\Theta - \Theta_m).$$

The Richardson number, which coincides with the inverse of the Froude number squared and measures the ratio of potential over kinetic energy, is assumed to be large enough so that density variations are not negligible in the gravity term. The Boussinesq system (1.1) can be rigorously obtained from the compressible Navier-Stokes-Fourier equations in the zero Mach and Froude numbers limit [58]. An important feature of this model, particularly for numerical purposes, is that it avoids the calculation of sound waves [59].

In two dimensions, Boussinesq equations can be rewritten by introducing the vorticity of the fluid $\omega = \nabla^\perp \cdot u = \partial_{x_1} u_2 - \partial_{x_2} u_1$,

$$\omega_t + u \cdot \nabla \omega = \frac{1}{\mathrm{Re}} \Delta \omega + \partial_{x_1} \theta,$$
$$\theta_t + u \cdot \nabla \theta = \frac{1}{\mathrm{Pe}} \Delta \theta,$$

where the velocity can be recovered from the vorticity by the Biot-Savart law

$$u = -\nabla^\perp (-\Delta)^{-1} \omega = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \star \omega.$$

For inviscid and non diffusive fluids, i.e., $1/\mathrm{Re} = 1/\mathrm{Pe} = 0$, the equations can be formally identified with the 3d Euler equations in vorticity form for axisymmetric swirling flows away from the axis [91]. It is well-known that the global regularity of these equations is still an outstanding open problem.

In this dissertation, we will consider two-dimensional viscous flows with infinite Péclet number, so our equations will read as follows

$$\begin{cases} \nabla \cdot u = 0, \\ D_t u - \Delta u = -\nabla P + \theta e_2, \\ D_t \theta = 0, \end{cases}$$

where the notation $D_t = \partial_t + u \cdot \nabla$ for the total or *material* derivative will be used throughout this work. In chapter two, we will consider initial temperature given by the characteristic function of a bounded domain, what we will simply call a *patch*,

$$\theta_0(x) = \begin{cases} \theta_0 & x \in D_0, \\ 0 & x \notin D_0. \end{cases}$$

We will first introduce the mathematical state of the art of the 2d Boussinesq equations and then we will study the *Boussinesq temperature patch problem.*

1.2 Inhomogeneous Navier-Stokes

Classic fluid mechanics equations arise from the fundamental physical laws of mass and momentum conservation under the continuum assumption. Euler (1757) was the first to describe the general motion of *ideal* fluids through a partial differential system. Mass conservation is there written as a continuity equation for the density, ρ, of the fluid moving with velocity u,

$$\rho_t + \nabla \cdot (\rho u) = 0,$$

while the conservation of momentum reads as follows

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = \rho f - \nabla p.$$

Above, f denotes body forces, such as gravity, and p denotes the pressure, which only acts in the normal direction. Friction forces are thus not considered in Euler equations. Euler already noted that, to close the system, an *equation of state*, usually relating the pressure with the density, is then needed. Assuming the compressibility effects are negligible, as is the case in liquids and low Mach number flows, the incompressibility condition is added

$$\nabla \cdot u = 0.$$

Considering no body forces for simplicity, $f = 0$, incompressible Euler system can be then rewritten as

$$\begin{cases} \nabla \cdot u = 0, \\ \rho u_t + \rho u \cdot \nabla u = -\nabla p \\ \rho_t + u \cdot \nabla \rho = 0. \end{cases}$$

Navier (1822) started the study of frictional forces in fluids and later Stokes (1845) completed the model for the viscous stress tensor that gives rise to the widely-known Navier-Stokes equations. Two hypothesis are made: a linear relationship between the viscous stress tensor and the rate-of-strain tensor (the so-called Newtonian fluids) and the isotropic condition, that is, the fluid properties does not depend on the spatial direction. Then, the momentum Navier-Stokes equations are

$$\rho u_t + \rho u \cdot \nabla u = -\nabla p + \nabla \cdot (\lambda(\nabla \cdot u)\mathbb{I} + \mu\mathbb{D}),$$

where μ is called the dynamic viscosity, λ denotes the bulk viscosity and $\mathbb{D}u$ denotes the symmetric part of the gradient

$$\mathbb{D}u = \nabla u + \nabla u^*.$$

Since initially homogeneous fluids, $\rho_0(x) = \rho_0$, remain homogeneous for all time in incompressible flows, *incompressible Navier-Stokes equations* usually refer to the system governing a constant density fluid with constant viscosity,

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u = -\nabla p + \mu \Delta u. \end{cases}$$

However, incompressible flows of non-homogeneous fluids evolve according to the inhomogeneous Navier-Stokes equations

$$\begin{cases} \nabla \cdot u = 0, \\ \rho_t + u \cdot \nabla \rho = 0, \\ \rho(u_t + u \cdot \nabla u) = \nabla \cdot (\mu \mathbb{D} u - p\, \mathbb{I}). \end{cases}$$

Due to salinity and stratification, these equations are widely used to model oceanic or river currents. They are also useful to describe mixtures of non-reactant liquids. Viscosity is thus sometimes considered to be density-dependent $\mu = \mu(\rho)$.

We will consider these equations on the plane and with constant positive viscosity. To simplify notation, we take $\mu = 1$, so the system will be given by

$$\begin{cases} \nabla \cdot u = 0, \\ D_t \rho = 0, \\ \rho D_t u = \Delta u - \nabla p. \end{cases}$$

In particular, we are interested in the so-called *density patches* solutions, that is to say, the initial density will be given by

$$\rho_0(x) = \begin{cases} \rho_1 & x \in D_0, \\ \rho_2 & x \in D_0^c = \mathbb{R}^2 \setminus D_0. \end{cases}$$

The patch will then be transported by the fluid flow

$$X(x,t) = x + \int_0^t u(\tau, X(\tau, x)) d\tau.$$

In the third chapter we will first comment some known results and properties of inhomogeneous Navier-Stokes equations and later we will focus on Lions' density patch problem.

1.3 Incompressible Porous Media

The dynamics of incompressible flows through a porous medium in $\mathbb{R}^d$ are governed by the following equations

$$\begin{cases} \nabla \cdot u = 0, \\ \rho_t + u \cdot \nabla \rho = 0, \\ \frac{\mu}{\kappa} u = -\nabla p - g \rho e_d, \end{cases}$$

where ρ is the density of the fluid, u is the velocity and p is the pressure. The symbols μ, κ and g denotes the viscosity of the fluid, the permeability of the medium and the gravity, respectively.

In the momentum equation, the velocity, instead of the acceleration, is proportional to the gradient of the pressure and external forces. This law, first determined by Darcy [50] based on physical experiments, can also be deduced from Stokes equations using homogenization [113]. The basic idea is that the porosity of the medium restrains the fluid motion, so that the inertia terms become negligible and the viscosity force acts as a restoring force linearly with the velocity, the permeability being the proportionality constant.

In this work, we are interested in a particular setting known as the *Muskat problem*. It describes the evolution of the free interface between two incompressible and immiscible fluids moving in a porous media. We will consider an isotropic medium, i.e., constant permeability, and constant but different densities and viscosities for each fluid

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in \mathbb{R}^2 \setminus D^1(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in \mathbb{R}^2 \setminus D^1(t). \end{cases} \tag{1.2}$$

A detailed discussion on the known results, properties and formulation of the Muskat problem is given in the fourth and last chapter.

Chapter 2

Boussinesq temperature patch problem

2.1 Introduction

In this chapter we consider the following active scalar equation

$$\theta_t + u \cdot \nabla\theta = 0, \tag{2.1}$$

for incompressible fluids

$$\nabla \cdot u = 0, \tag{2.2}$$

which depending on the physical context can be seen as the mass or energy conservation. In this latter case $\theta = \theta(x,t)$ corresponds to the temperature transported without diffusion by the fluid, which moves with velocity $u = (u_1(x,t), u_2(x,t))$. We close the system with the Boussinesq model for the momentum equation

$$u_t + u \cdot \nabla u - \nu\Delta u + \nabla P = g(0,\theta), \tag{2.3}$$

where P is the pressure, $\nu > 0$ the viscosity, g the gravity, $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$. By a change of variables we simplify matters by taking gravity $g = 1$ and viscosity $\nu = 1$.

This system first arose as a model to study natural convection phenomena in geophysics [65], [92], as for example in the very important Rayleigh-Bénard problem [28]. There, the density variations can frequently be neglected except in the buoyancy term, avoiding in this way the calculation of sound waves. From the mathematical point of view, the interest resides on its connection to the Navier-Stokes and Euler equations since it presents vortex stretching [91].

For that reason, the well-posedness of this model has recently attracted a lot of attention, starting with the results of Chae [21] and Hou and Li [78] for regular initial data in the whole space $\mathbb{R}^2$. Later, making use of paradifferential calculus techniques, results for rougher initial data appeared. In particular, Abidi and Hmidi [1] established the global well-posedness for initial data in the Besov space $B^0_{2,1}$ (see (2.45) for definition), then Hmidi and Keraani [73] proved global existence and regularity for initial data $\theta_0 \in L^2$, $u_0 \in H^s$, $s \in [0,2)$ and finally

Danchin and Paicu obtained the uniqueness [44]. The persistence of regularity in Sobolev spaces was completed by Hu, Kukavica and Ziane in [79]. Available global-in-time results in three dimensions require the initial data to be small [44], as the system includes Navier-Stokes equations as a particular case.

The Boussinesq system is also used to model large scale atmospheric and oceanic flows, where the viscosity and diffusivity constants are usually different in the horizontal and vertical directions. In these situations there are similar results for the case with anisotropic and partial dissipation [2], [14], [84], [3], [85], and positive diffusivity but no viscosity [21], [74], including results with Yudovich type initial data [45]. In contrast, the global well-posedness of the full inviscid case remains still as an open question, mathematically analogous indeed to the incompressible axi-symmetric swirling three-dimensional Euler equations [91]. Simulations first indicated the possibility of finite-time blow-up, but there were also numerical evidence in the opposite direction [52]. Recently, based on numerical studies, a new scenario for finite time blow-up in 3D Euler equations has been proposed [90]. This situation has been adapted to rigorously prove the existence of finite time blow-up first for a 1D model of the Boussinesq equations [26] and more recently for a modified version of the two dimensional case including incompressibility [75], [81]. In [53], the authors are able to prove finite-time singularity formation in 2D Boussinesq for finite-energy initial data on spatial domains that can get arbitrarily close to the half-space.

Related to these problems, we consider here a case with singular initial data: the so-called *Boussinesq temperature patch problem* for (2.1)-(2.3). From (2.1) and the definition of the particle trajectories,

$$\begin{cases} \dfrac{dX}{dt}(x,t) = u(X(x,t),t), \\ X(x,0) = x, \end{cases} \tag{2.4}$$

an initial temperature patch $\theta_0 = 1_{D_0}$, i.e., the characteristic function of a simply connected bounded domain D_0, will remain as a patch $\theta(t) = 1_{D(t)}$, where $D(t) = X(D_0, t)$. Therefore, one may wonder whether its boundary preserves the initial regularity. This kind of problems were studied in the 80s for the well-known vortex patch problem. In that case, it was first thought from numerical results that there was finite-time blow-up, but later the global regularity was proved by Chemin [22] using paradifferential calculus and striated regularity techniques. The same result was proved by Bertozzi and Constantin [11] in a geometrical way making use of harmonic analysis tools.

Counter dynamics scenarios to look for singularity formation come from the evolution of the interface between fluids of different characteristics. Used to model physically important problems such as water waves, porous media, inhomogeneous flow or frontogenesis, this contour dynamics setting has attracted a lot of attention in the last years. The appearance of finite time singularities was first proved for the Muskat problem [16], [20], Euler [17] and Navier-Stokes equations [18] and from there different scenarios and results appeared [40], [41]. In the SQG active scalar incompressible system [27], for the patch problem [103] there is numerical evidence of pointwise collapse with curvature blow-up [32]. In addition, it has been shown that the control of the curvature removes the possibility of pointwise interface collapse [60].

Recently Danchin and Zhang proved [48] that if the initial temperature of the Boussinesq system (2.1)-(2.3) is a $C^{1+\gamma}$ patch, it remains with the same regularity for all time. In this sense, we will denote $\partial D(t) \in C^{1+\gamma}$ if there exists a parametrization of the boundary

$$\partial D(t) = \left\{ z(\alpha, t) \in \mathbb{R}^2, \alpha \in [0,1] \right\} \tag{2.5}$$

with $z(t) \in C^{1+\gamma}$.

From the above, one may wonder if the curvature of a temperature patch in the Boussinesq system can blow up without self-intersection or if, on the contrary, it remains bounded for all time. We prove here that the latter occurs, that is, we show the persistence of $W^{2,\infty}$ regularity for Boussinesq temperature patches.

In [48], the authors use Besov spaces to measure regularity and the techniques of striated regularity to get the $C^{1+\gamma}$ propagation. The main idea is that to control the Hölder regularity of the patch one just needs to control the gradient of the velocity in certain directions, which can be translated into the vorticity equation and then treat this as a forced heat equation, hence achieving the gain of two derivatives. This result was done for a general $\theta_0 \in B^{2/q-1}_{q,1}$, $q \in (1,2)$ and then applied to temperature patches, as a patch always belongs to that space.

In this thesis we exploit the fact of θ being a patch. Indeed, one can get the Hölder persistence of regularity by controlling the $L^1(0,T;C^\gamma)$ norm of the gradient of the velocity using the particle trajectories of the system. This can be seen from the vorticity equation rewritten as

$$\omega(t) = e^{t\Delta}\omega_0 - (\partial_t - \Delta)_0^{-1} \nabla \cdot (u\omega) + (\partial_t - \Delta)_0^{-1} \partial_1 \theta, \tag{2.6}$$

where $(\partial_t - \Delta)_0^{-1} f$ denotes the solution of the heat equation with force f and zero initial condition:

$$(\partial_t - \Delta)_0^{-1} f := \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau.$$

Above we use the standard notation $e^{t\Delta} f = \mathcal{F}^{-1}(e^{-t|\xi|^2}\hat{f})$, where $\hat{}$ and $\mathcal{F}^{-1}$ denote Fourier transform and its inverse.

We note then that one could get $u \in L^1(0,T;B^2_{\infty,\infty})$ by choosing more regular initial conditions, since the main limitation comes from the temperature term and singular integrals are bounded on Besov spaces. However, to control the boundedness of the curvature of the patch it will be needed to control the L^∞ norm of the second derivatives of the velocity,

$$\partial_k \partial_j u_i(t) = \partial_i^\perp \partial_j (-\Delta)^{-1} \left(e^{t\Delta} \partial_k \omega_0 - (\partial_t - \Delta)_0^{-1} \partial_k \nabla \cdot (u\omega) + (\partial_t - \Delta)_0^{-1} \partial_k \partial_1 \theta \right),$$

$(\partial_1^\perp, \partial_2^\perp) = (-\partial_2, \partial_1)$, i.e., we will need $u \in L^1(0,T;W^{2,\infty})$. This is not trivial since neither the Riesz transforms nor the operators $\partial_i \partial_j (\partial_t - \Delta)_0^{-1}$ are bounded for a general function in L^∞. In fact, it is known that this operator takes bounded functions to BMO [106] (defined using parabolic cylinders instead of Euclidean balls). In addition, while one may expect to get rid of the Riesz transform by using striated regularity techniques (assuming that one could first get further regularity in the tangential direction to interpolate the L^∞ norm), it would still be necessary to bound $\Delta(\partial_t - \Delta)_0^{-1}\theta$ (see Remark 2.3.3). In fact, the associated kernel

$$\frac{1}{4\pi t^2} \left(\frac{|x|^2}{4t} - 1 \right) e^{-|x|^2/4t} \tag{2.7}$$

is not integrable due to the singularity at the origin along parabolas $t = c|x|^2$. However, the cancellation due to the sign change through the parabola $t = |x|^2/4$ allows to appropriately define these operators as principal values. What is more, although this kernel has nonzero mean, we will show that it is bounded for θ a patch. First we give a $C^{1+\gamma}$ result. Later we refine the idea used in [11] to bound the gradient of the velocity, i.e., the combined fact that for a $C^{1+\gamma}$ patch the intersection of a small circle with its boundary is *almost* a semicircle and that the kernel was even with zero mean on circles. Although the latter is not true in our case (see kernel (2.7)), we encounter that the kernels present certain time-space cancellations on circles. In this scenario, the kernels now depend also on time so that the picture is no more static and therefore we need to take care of the evolution of the distance of the point to the boundary.

The above result would prove that for θ a patch $\nabla\omega$ is bounded. We can polish the idea and adjust it to the operators

$$\partial_i^{\perp}\partial_j(-\Delta)^{-1}\partial_k\partial_1(\partial_t-\Delta)_0^{-1}.$$

The strategies above allow us to control the particle trajectories by estimating all the second derivatives of the velocity. There is hope that one can take advantage of the cancellation $W \cdot \nabla\theta \equiv 0$, where W is a vector field tangent to the patch [48]. Going beyond, we can take advantage of this cancellation and the $W^{2,\infty}$ result to prove the persistence of regularity for $C^{2+\gamma}$ patches.

In relation to the initial conditions, for $\omega_0 \in H^s$ it is not possible to proceed as before for $s = 0$ due to the term ω_1 in (2.6): specifically, one encounters the failure of the embedding of $H^1(\mathbb{R}^2)$ into L^∞ and the restriction from the maximum gain of the heat equation shown in [56]. Nevertheless, if we don't restrict ourselves to Sobolev spaces, the result of persistence of regularity for the patch is still true for more general initial conditions (see Section 2.5). In this sense, the initial conditions to achieve the $C^{1+\gamma}$ result are at the same level of regularity to those in [48] but in the Sobolev space scale.

The chapter is organized as follows: In the next section, we give a preliminary result of the $C^{1+\gamma}$ regularity using particle trajectories and Sobolev spaces for the velocity. Then, in Section 2.3 we present the main result: the control of the curvature. First we define the operators involved, then show the new cancellations encountered and finally prove that $u \in L^1(0,T;W^{2,\infty})$. In Section 2.4 we use the extra regularity of the patch in the tangential direction to show the persistence of $C^{2+\gamma}$ regularity. Additionally, in Section 2.5 we give results for initial conditions in different spaces.

2.2 Persistence of $C^{1+\gamma}$ regularity

In this section the persistence of $C^{1+\gamma}$ regularity is proved. We state the result for velocity fields belonging to the Sobolev space $H^{\gamma+s}$, $s \in (0, 1-\gamma)$ (see Section 2.5 for $u_0 \in B^{\gamma-1+s_1}_{\infty,\infty} \cap H^{s_2}$, $s_1 \in (0, 1-\gamma)$, $s_2 \in (0,1)$). This is roughly at the same level of regularity needed for the result in [48], where for a general initial vorticity the striated regularity condition can be translated into $u_0 \in B^{\gamma-1}_{\infty,\infty} \cap B^{-1+2/q}_{q,1} \hookrightarrow B^{\gamma-1}_{\infty,\infty} \cap B^0_{2,1}$, with $q \in (1, \frac{2}{2-\gamma})$.

Theorem 2.2.1. *Assume $\gamma \in (0,1)$, $s \in (0,1-\gamma)$. Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in C^{1+\gamma}$, $u_0 \in H^{\gamma+s}$ a divergence-free vector field and $\theta_0 = 1_{D_0}$ the characteristic function of D_0. Then, there is a unique global solution (u,θ) of* (2.1)-(2.3) *such that*

$$\theta(x,t) = 1_{D(t)}(x) \text{ and } \partial D \in L^\infty(0,T;C^{1+\gamma}),$$

where $D(t) = X(D_0,t)$ with X the particle trajectories associated to the velocity field. Moreover,

$$u \in L^\infty(0,T;H^{\gamma+s}) \cap L^2(0,T;H^{1+\gamma+s}) \cap L^1(0,T;H^{2+\mu}) \cap L^1(0,T;C^{1+\gamma+\tilde{s}}),$$

for any $T > 0$, $\mu < \min\{\frac{1}{2}, \gamma+s\}$, $0 < \tilde{s} < s$.

Proof: The first part of the proof consists of *a priori* estimates. From the transport equation for the temperature one gets

$$\|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2}, \qquad \|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

The basic energy inequality also holds

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq \|\theta_0\|_{L^2}\|u\|_{L^2}, \tag{2.8}$$

so that by Grönwall's lemma

$$\|u(t)\|_{L^2}^2 \leq \left(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2\right) e^t - \|\theta_0\|_{L^2}^2, \tag{2.9}$$

$$\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2\right) e^t. \tag{2.10}$$

Now, we apply the operator $\Lambda^{\gamma+s}$ to the velocity equation to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^{\gamma+s}u\|_{L^2}^2 + \|\Lambda^{1+\gamma+s}u\|_{L^2}^2 &= -\int \Lambda^{1+\gamma+s}u \cdot \Lambda^{-1+\gamma+s}(u\cdot\nabla u) + \int \Lambda^{2(\gamma+s)}u_2\theta \\ &\leq \|\Lambda^{1+\gamma+s}u\|_{L^2}\|\Lambda^{-1+\gamma+s}(u\cdot\nabla u)\|_{L^2} + \|\Lambda^{2(\gamma+s)}u\|_{L^2}\|\theta\|_{L^2}.\end{aligned}$$

Adding the energy inequality (2.8) to the above one and noting that $2(s+\gamma) < s+\gamma+1$ for $s+\gamma \in (0,1)$ leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{H^{\gamma+s}}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{H^{1+\gamma+s}}^2 \leq \|u\|_{L^2}^2 + \|\theta_0\|_{L^2}\|u\|_{H^{1+\gamma+s}} \\ + \|\Lambda^{1+\gamma+s}u\|_{L^2}\|\Lambda^{-1+\gamma+s}(u\cdot\nabla u)\|_{L^2}.\end{aligned}$$

By Sobolev embeddings and Hölder's inequality the last term above can be bounded as

$$\|\Lambda^{-1+\gamma+s}(u\cdot\nabla u)\|_{L^2} \leq c\|u\cdot\nabla u\|_{L^{2/(2-\gamma-s)}} \leq c\|u\|_{L^{2/(1-\gamma-s)}}\|\nabla u\|_{L^2} \leq c\|\Lambda^{\gamma+s}u\|_{L^2}\|\nabla u\|_{L^2},$$

so by Young's inequality and (2.9) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^{\gamma+s}}^2 + \frac{1}{2}\|u\|_{H^{1+\gamma+s}}^2 \leq \left(\|\theta_0\|_{L^2}^2 + \|u_0\|_{L^2}^2\right) e^t + c\|\nabla u\|_{L^2}^2\|u\|_{H^{\gamma+s}}^2. \tag{2.11}$$

To conclude, from (2.10) and Grönwall's inequality applied to (2.11) it follows that

$$\|u(t)\|^2_{H^{\gamma+s}} \leq c_1(\|u_0\|_{L^2}, \|u_0\|_{H^{\gamma+s}}, \|\theta_0\|_{L^2}, t),$$

$$\int_0^t \|u(\tau)\|^2_{H^{1+\gamma+s}} d\tau \leq c_2(\|u_0\|_{L^2}, \|u_0\|_{H^{\gamma+s}}, \|\theta_0\|_{L^2}, t).$$

These estimates can be justified by the usual limiting procedure [91].

Finally, to show uniqueness in this class, we consider two different solutions (u_1, θ_1), (u_2, θ_2) with the same initial data, denote the difference $\tilde{u} = u_2 - u_1$, $\tilde{\theta} = \theta_2 - \theta_1$ and take inner product:

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2_{L^2} \leq \|\nabla u_2\|_{L^\infty}\|\tilde{u}\|^2_{L^2} - \|\nabla\tilde{u}\|^2_{L^2} + \int \tilde{u}_2\tilde{\theta}dx$$
$$\leq \|\nabla u_2\|_{L^\infty}\|\tilde{u}\|^2_{L^2} - \|\nabla\tilde{u}\|^2_{L^2} - \int \nabla\tilde{u}_2 \cdot \nabla\Delta^{-1}\tilde{\theta}dx \leq \|\nabla u_2\|_{L^\infty}\|\tilde{u}\|^2_{L^2} + 2\|\nabla\Delta^{-1}\tilde{\theta}\|^2_{L^2},$$

where the incompressibility condition of u_1 and u_2 and integration by parts have been repeatedly used. As $\|\nabla\Delta^{-1}\theta\|_{L^2} = \|\Lambda^{-1}\theta\|_{L^2}$, proceeding as above

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}\tilde{\theta}\|^2_{L^2} = \int(\Delta^{-1}\tilde{\theta})\nabla\cdot(u_1\tilde{\theta})dx + \int(\Delta^{-1}\tilde{\theta})\nabla\cdot(\tilde{u}\theta_2)dx$$
$$= -\int \nabla\Delta^{-1}\tilde{\theta}\cdot\nabla u_1\cdot\nabla\Delta^{-1}\tilde{\theta} - \int \nabla\Delta^{-1}\tilde{\theta}\cdot\tilde{u}\theta_2 dx$$
$$\leq \|\nabla\Delta^{-1}\tilde{\theta}\|^2_{L^2}\|\nabla u_1\|_{L^\infty} + \frac{1}{2}\|\theta_2\|_{L^\infty}\|\tilde{u}\|^2_{L^2} + \frac{1}{2}\|\theta_2\|_{L^\infty}\|\nabla\Delta^{-1}\tilde{\theta}\|^2_{L^2}.$$

From the sum of both inequalities and Grönwall's inequality the uniqueness follows.

Now the regularity of the velocity will be improved by using the vorticity equation. We will show that $u \in L^1(0,T;H^2) \cap L^1(0,T;C^{1+\gamma+\tilde{s}})$, $0 < \tilde{s} < s$. We consider the vorticity equation as a forced heat equation

$$\omega_t - \Delta\omega = -u\cdot\nabla\omega + \partial_{x_1}\theta, \quad \omega|_{t=0} = \omega_0,$$

and split the solution into three parts

$$\omega(t) = \omega_1 + \omega_2 + \omega_3, \quad \omega_1 = e^{t\Delta}\omega_0, \quad \omega_2 = -(\partial_t - \Delta)^{-1}_0\nabla\cdot(u\omega), \quad \omega_3 = (\partial_t - \Delta)^{-1}_0\partial_1\theta. \tag{2.12}$$

Since $\omega_0 \in H^{-1+\gamma+s}$, by standard estimates for solutions of the heat equation in Sobolev spaces, we deduce that $\omega_1 \in L^1(0,T;H^{1+\gamma+\tilde{s}})$ for $0 < \tilde{s} < s$, and therefore $\omega_1 \in L^1(0,T;C^{\gamma+\tilde{s}})$ by Sobolev embedding. This is sharp as in general $\omega_1 \notin L^1(0,T;H^{1+\gamma+s})$ [56].

For the second part ω_2 it suffices to prove that $u\omega \in L^q(0,T;H^{\gamma+s})$ for some $q > 1$. In that case, one can make use of maximal regularity results for the heat equation to conclude that $\omega_2 \in L^q(0,T;H^{1+\gamma+s})$ and hence $\omega_2 \in L^1(0,T;C^{\gamma+s})$ (see for example [56]). From the fact that $u \in L^\infty(0,T;H^{\gamma+s}) \cap L^2(0,T;H^{1+\gamma+s})$, by interpolation it follows that,

$$u \in L^\rho(0,T;H^{1+r}), \quad \rho = \frac{2}{1-(s+\gamma)+r} > 2, \quad 0 < r < s+\gamma.$$

Since $\omega \in L^2(0,T;H^{\gamma+s})$ and $\|u\omega\|_{H^{\gamma+s}} \le c\|u\|_{H^{1+r}}\|\omega\|_{H^{\gamma+s}}$, by Hölder's inequality it follows that

$$u\omega \in L^q(0,T;H^{\gamma+s}), \quad \frac{1}{q} = \frac{1}{\rho} + \frac{1}{2} < 1.$$

The temperature term, ω_3, must be treated in a different way, as up to now $\theta(t)$ only belongs to $L^2(\mathbb{R}^2)$, so the best we can achieve in the scale of Sobolev spaces is $\omega_3 \in L^\infty(0,T;H^1)$. Anyway, even if we assume that θ remains as a patch, it would belong to H^σ, $\sigma < 1/2$ (as the characteristic function of a set with regular boundary belongs to these Sobolev spaces, see e.g. [96]), but $\omega_3 \in L^\infty(0,T;H^{1+\sigma})$ is not enough if $\gamma \ge 1/2$. We use instead that $\theta \in L^\infty(0,T;L^\infty) \hookrightarrow L^\infty(0,T;B^0_{\infty,\infty})$. Then $\omega_3 \in L^\infty(0,T;B^1_{\infty,\infty})$. Since $B^1_{\infty,\infty} \hookrightarrow C^r$ $\forall r < 1$, summing up the three terms and recalling that the Riesz transforms are continuous on Sobolev and Hölder spaces, we finally get the result $u \in L^1(0,T;H^2) \cap L^1(0,T;C^{1+\gamma+\tilde{s}})$.

Remark 2.2.2. *By interpolation, we could improve the time integrability to get $u \in L^p(0,T;H^2)$ for some $p = p(\gamma,s,\tilde{s}) \in (1,2)$, although we won't use it. However, to improve the spatial regularity, the restriction comes from the temperature term. Despite $\theta_0 \in H^\sigma$ for any $\sigma < 1/2$, results in [79] are not sufficient to ensure that θ remains in H^σ, due to the fact that the initial velocity has a low regularity comparable to that of the initial temperature. Once we prove that the patch is preserved and remains regular, the lines above would immediately imply that $u \in L^1(0,T;H^{2+\mu})$, $\mu < \{1/2, \gamma+s\}$.*

To conclude, we note that once chosen the parametrization (2.5), the tangent vector of the patch is given by

$$\partial_\alpha z(\alpha,t) = \nabla X(z_0(\alpha),t) \cdot \partial_\alpha z_0(\alpha). \tag{2.13}$$

From (2.4), one obtains that

$$\partial_t \|\nabla X\|_{C^\gamma} \le \|\nabla u\|_{L^\infty}\|\nabla X\|_{C^\gamma} + \|\nabla X\|^{1+\gamma}_{L^\infty}\|\nabla u\|_{C^\gamma},$$

so, since $\nabla u \in L^1(0,T;C^\gamma)$, it follows from Grönwall's lemma that

$$\|\nabla X\|_{C^\gamma} \le \|\nabla X_0\|_{C^\gamma} e^{\int_0^t \|\nabla u\|_{L^\infty} d\tau} + \int_0^t \|\nabla u(\tau)\|_{C^\gamma}\|\nabla X(\tau)\|^{1+\gamma}_{L^\infty} e^{\int_\tau^t \|\nabla u\|_{L^\infty} ds},$$

and hence $\|z\|_{L^\infty(0,T;C^{1+\gamma})} \le C(T)$. □

2.3 Control of curvature

This section deals with the main result of this chapter: the control of the curvature of the patch. To that end we bound $\nabla^2 u$ in $L^1(0,T;L^\infty)$, which requires proving that the operators $\partial_k\partial_1(\partial_t - \Delta)_0^{-1}$ and $\partial_i^\perp\partial_j(-\Delta)^{-1}\partial_k\partial_1(\partial_t-\Delta)_0^{-1}$ applied to a patch are bounded.

Theorem 2.3.1. *Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in W^{2,\infty}$, $u_0 \in H^{1+s}$ with $s \in (0,1/2)$ a divergence-free vector field and $\theta_0 = 1_{D_0}$ the* *characteristic function of* D_0. *Then, there exists a unique global solution* (u, θ) *of* (2.1)-(2.3) *such that*

$$\theta(x,t) = 1_{D(t)}(x) \text{ and } \partial D \in L^\infty(0,T;W^{2,\infty})$$

where $D(t) = X(D_0, t)$. *Moreover,*

$$u \in C(\mathbb{R}_+; H^{1+s}) \cap L^2(0,T;H^{2+s}) \cap L^p(0,T;H^{2+\mu}) \cap L^q(0,T;W^{2,\infty}),$$

for any $T > 0$, $\mu < \frac{1}{2}$, $1 \le p < 2/(1+\mu-s)$, $1 \le q < 2/(2-s)$.

Proof: From the previous theorem θ remains as a $C^{1+\gamma}$ patch. Since the characteristic function of a set with regular boundary belongs to H^μ for any $\mu < 1/2$, the result in [79] yields the existence and uniqueness of solutions $u \in C(\mathbb{R}_+; H^{1+s}) \cap L^2(0,T;H^{2+s})$, $\theta \in C(\mathbb{R}_+; H^s)$.

Proceeding as in the second part of the previous theorem, we split the vorticity as in (2.12). First, as $u \in L^\infty(0,T;H^{1+s})$ and $\omega \in L^2(0,T;H^{1+s})$ it follows that $u\omega \in L^2(0,T;H^{1+s})$. Using this, by the properties of the heat kernel the first two parts are treated as before:

$$\omega_0 \in H^s \Longrightarrow \omega_1 \in L^1(0,T;H^{2+\tilde{s}}), \tilde{s} \in (0,s), \tag{2.14}$$

$$u\omega \in L^2(0,T;H^{1+s}) \Longrightarrow \omega_2 \in L^2(0,T;H^{2+s}). \tag{2.15}$$

As the third term remains the same, $\omega_3 \in L^\infty(0,T;H^{1+\mu})$, one obtains by interpolation $u \in L^p(0,T;H^{2+\mu})$ where $p = \frac{2(1-(s-\tilde{s}))}{1+\mu-s-(s-\tilde{s})} > \frac{4}{3}$.

We are thus left to prove that $u \in L^1(0,T;W^{2,\infty})$, as the persistence of regularity for the patch will follow by applying Grönwall's inequality to the equation satisfied by the second derivatives of the particle trajectories:

$$\frac{d}{dt}\nabla^2 X = \nabla X^T \cdot \nabla^2 u(X) \cdot \nabla X + \nabla^2 X \cdot \nabla u(X).$$

In order to prove that $\nabla^2 u \in L^1(0,T;L^\infty)$, we use the Biot-Savart formula

$$\left(\nabla^2 u\right)_{ijk} = \partial_k \partial_j u_i = -\partial_k(-\Delta)^{-1}\partial_i^\perp \partial_j(\omega_1+\omega_2+\omega_3),$$

where $i, j, k \in \{1,2\}$ and $\partial_1^\perp = -\partial_2$, $\partial_2^\perp = \partial_1$. We denote

$$\begin{aligned}
(\nabla^2 v_1)_{ijk} &= -\partial_k(-\Delta)^{-1}\partial_i^\perp\partial_j\omega_1 = -\partial_k(-\Delta)^{-1}\partial_i^\perp\partial_j e^{t\Delta}\omega_0,\\
(\nabla^2 v_2)_{ijk} &= -\partial_k(-\Delta)^{-1}\partial_i^\perp\partial_j\omega_2 = \partial_k(-\Delta)^{-1}\partial_i^\perp\partial_j(\partial_t-\Delta)_0^{-1}\nabla\cdot(u\omega),\\
(\nabla^2 v_3)_{ijk} &= -\partial_k(-\Delta)^{-1}\partial_i^\perp\partial_j\omega_3 = -\partial_k(-\Delta)^{-1}\partial_i^\perp\partial_j(\partial_t-\Delta)_0^{-1}\partial_1\theta.
\end{aligned} \tag{2.16}$$

As $\omega_1 \in L^1(0,T;H^{2+\tilde{s}})$ and $\omega_2 \in L^2(0,T;H^{2+s})$, it is clear that $\nabla^2 v_1, \nabla^2 v_2 \in L^1(0,T;L^\infty)$.

The temperature term can be written as an operator applied to θ:

$$\left(\nabla^2 v_3\right)_{ijk} = R_i R_j \partial_k \partial_1 (\partial_t-\Delta)_0^{-1}\theta \tag{2.17}$$

where R_i, R_j denote Riesz transforms.

The rest of the proof is structured as follows: first, in Section 2.3.1 we define the operators $\partial_k\partial_1\left(\partial_t-\Delta\right)_0^{-1}$ and $R_iR_j\partial_k\partial_1\left(\partial_t-\Delta\right)_0^{-1}$. Later in Section 2.3.2 we proceed to bound these operators applied to the patch, i.e., we compute the bounds in $L^1(0,T;L^\infty)$ for the gradient of the vorticity and the second derivatives of the velocity. The bounds found give $\nabla^2 u \in L^q(0,T;L^\infty)$, $1 \le q < 2/(2-s)$ (see Remark 2.3.2).

2.3.1 Definition of the operators

Singular heat kernels: $\partial_k\partial_1(\partial_t-\Delta)_0^{-1}$

Denote the heat kernel $K(x,t) = \frac{1}{4\pi t}e^{-|x|^2/4t}$. Then, the operators for the patch can be written as

$$\partial_k\omega_3(x,t) = \partial_k\partial_1(\partial_t-\Delta)_0^{-1}\theta(x,t) := \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\mathbb{R}^2}(\partial_k\partial_1 K)(x-y,t-\tau)\theta(y,\tau)dyd\tau, \quad (2.18)$$

where

$$\begin{cases} \partial_1^2 K(x,t) = \dfrac{1}{8\pi t^2}\left(\dfrac{x_1^2}{2t}-1\right)e^{-|x|^2/4t}, \\ \partial_2\partial_1 K(x,t) = \dfrac{x_1x_2}{16\pi t^3}e^{-|x|^2/4t}. \end{cases} \quad (2.19)$$

Note that the principal value is needed: due to the singularity of the kernels at $(x,t) = (0,0)$ along parabolas $\tau = Mr^2$ ($M \neq 1/4$ for $\partial_1^2 K$), the kernels are not absolutely integrable:

$$\int_0^t\int_{\mathbb{R}^2}|\partial_1^2 K(y,\tau)|dyd\tau = +\infty.$$

It is clear then that these operators are not bounded for a general L^∞ function.

Here we point out that while $\partial_1\partial_2 K(\cdot,t)$ clearly has zero mean on circles for any t, this is not true for $\partial_1^2 K$. However, the time integral provides a new kind of cancellation:

$$\begin{aligned} \lim_{\epsilon\to 0}\int_\epsilon^t\int_{B_R}\partial_1^2 K(y,\tau)dyd\tau &= \lim_{\epsilon\to 0}\int_\epsilon^t\int_0^R \frac{r}{4\tau^2}\left(\frac{r^2}{4\tau}-1\right)e^{-r^2/4\tau}d\tau dr \\ &= \lim_{\epsilon\to 0}\int_\epsilon^t \frac{-R^2e^{-R^2/4\tau}}{8\tau^2}d\tau = -\frac{1}{2}e^{-R^2/4t}, \end{aligned} \quad (2.20)$$

where B_R is the ball of radius R.

Oseen-type kernels: $R_iR_j\partial_k\partial_1(\partial_t-\Delta)_0^{-1}$

We will see that $\partial_k\omega_3 = \partial_k\partial_1(\partial_t-\Delta)_0^{-1}\theta$ is bounded if $\theta(t)$ is a $C^{1+\gamma}$ patch for all t. To get the boundedness of $R_iR_j\partial_k\partial_1(\partial_t-\Delta)_0^{-1}$ we need to treat the combined operators directly, which we will write as a convolution with an explicit kernel:

$$\begin{aligned} \left(\nabla^2 v_3\right)_{ijk}(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{y_i^\perp}{|y|^2}\partial_j\partial_k\omega_3(x-y,t)dy = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log|y|\partial_i^\perp\partial_j\partial_k\omega_3(x-y,t)dy \\ &= \partial_i^\perp\partial_j\partial_k\partial_1\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\mathbb{R}^2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}K(x-y-z,t-\tau)\log|y|dy\right)\theta(z,\tau)dzd\tau. \end{aligned}$$

We notice now that the term in brackets can be seen as the solution of the Laplace equation in $\mathbb{R}^2$ with force $K(x,t)$, which can be computed explicitly:

$$\Delta^{-1}K(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{4\pi t}e^{-\frac{|x-y|^2}{4t}}\log|y|dy = \frac{1}{2\pi}\left(\log|x| + \frac{1}{2}\int_{|x|^2/4t}^{\infty}\frac{e^{-z}}{z}dz\right).$$

Thus the operators can be written as

$$R_iR_j\partial_k\partial_1(\partial_t-\Delta)_0^{-1}\theta(x,t) = \partial_i^{\perp}\partial_j\partial_k\partial_1\int_0^t(\Delta^{-1}K)(t-\tau)*\theta(\tau)(x)d\tau,$$

or, as a kernel convolution in θ

$$R_iR_j\partial_k\partial_1(\partial_t-\Delta)_0^{-1}\theta(x,t) := \lim_{\epsilon\to 0}\int_0^{t-\epsilon}K_{ijk}(t-\tau)*\theta(\tau)(x)d\tau,$$

where

$$\begin{aligned} K_{ijk}(x,t) &:= \partial_1\partial_j\partial_i^{\perp}\partial_k(\Delta^{-1}K(x,t)) = \partial_1\partial_j\partial_i^{\perp}\partial_k\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{4\pi t}e^{-\frac{|x-y|^2}{4t}}\log|y|dy\right) \\ &= \frac{1}{2\pi}\partial_j\partial_i^{\perp}\partial_k\left(\frac{x_1}{|x|^2}\left(1-e^{-|x|^2/4t}\right)\right). \end{aligned} \tag{2.21}$$

The eight possible kernels reduce to four:

$$\begin{aligned} K_{111} &= \left(\frac{24x_1^3x_2}{\pi|x|^5} - \frac{12x_1x_2}{\pi|x|^3}\right)G(|x|,t) - \frac{e^{-|x|^2/4t}}{\pi(4t)^2}\left(\frac{12x_1^3x_2}{|x|^4} - \frac{6x_1x_2}{|x|^2} + \frac{4x_1^3x_2}{|x|^2}\frac{1}{4t}\right), \\ K_{112} &= \left(\frac{24x_1^2x_2^2}{\pi|x|^5} - \frac{3}{\pi|x|}\right)G(|x|,t) - \frac{e^{-|x|^2/4t}}{\pi(4t)^2}\left(-2 + \frac{4}{4t}\frac{x_1^2x_2^2}{|x|^2} + 12\frac{x_1^2x_2^2}{|x|^4}\right), \\ K_{122} &= \left(\frac{24x_1x_2^3}{\pi|x|^5} - \frac{12x_1x_2}{\pi|x|^3}\right)G(|x|,t) - \frac{e^{-|x|^2/4t}}{\pi(4t)^2}\left(\frac{12x_1x_2^3}{|x|^4} - \frac{6x_1x_2}{|x|^2} + \frac{4x_1x_2^3}{|x|^2}\frac{1}{4t}\right), \\ K_{211} &= \left(-\frac{24x_1^4}{\pi|x|^5} + \frac{24x_1^2}{\pi|x|^3} - \frac{3}{\pi|x|}\right)G(|x|,t) - \frac{e^{-|x|^2/4t}}{\pi(4t)^2}\left(\frac{12x_1^2}{|x|^2} - \frac{12x_1^4}{|x|^4} - \frac{4x_1^4}{|x|^2}\frac{1}{4t}\right), \\ K_{121} &= K_{112}, \quad K_{212} = -K_{111}, \quad K_{221} = -K_{111}, \quad K_{222} = -K_{112}, \end{aligned} \tag{2.22}$$

where

$$G(|x|,t) = \frac{1}{|x|^3}(1-e^{-|x|^2/4t}) - \frac{1}{4t|x|}e^{-|x|^2/4t}. \tag{2.23}$$

All these kernels are not integrable, but they show cancellations in circles. In fact, the kernels K_{111} and K_{122} have zero mean on circles, while for K_{112} and K_{211} we find the cancellation (2.20):

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{\epsilon}^{t}\int_{B_R}K_{112}(y,\tau)dyd\tau &= \lim_{\epsilon\to 0}\int_{\epsilon}^{t}\int_0^R\frac{r}{(4\tau)^2}\left(\frac{r^2}{4\tau}-1\right)e^{-r^2/4\tau}drd\tau = -\frac{1}{2}e^{-R^2/4t}, \\ \lim_{\epsilon\to 0}\int_{\epsilon}^{t}\int_{B_R}K_{211}(y,\tau)dyd\tau &= \lim_{\epsilon\to 0}\int_{\epsilon}^{t}\int_0^R\frac{3r}{(4\tau)^2}\left(1-\frac{r^2}{4\tau}\right)e^{-r^2/4\tau}drd\tau = -\frac{3}{2}e^{-R^2/4t}. \end{aligned} \tag{2.24}$$

We split these kernels in two parts, $K_{ijk} = K_{ijk}^* + K_{ijk}^o$, where $\int_{\partial B_R}K_{ijk}^o(y)d\sigma(y) = 0$.

2.3.2 Bound for $\nabla^2 u_3$

In this section we prove that the gradient of the vorticity and the second derivatives of the velocity are bounded in $L^1(0,T;L^\infty)$ if θ is a patch. First we prove that $\partial_k\omega_3 \in L^1(0,T;L^\infty)$. We do it carefully for $\partial_1\omega_3$ and then show the differences with $\partial_2\omega_3$.

Let $\gamma \in (0,1)$ and consider a parametrization $z(\cdot,t) \in C^{1,\gamma}$ as in (2.5), which we know that exists thanks to Theorem 2.2.1 and verifies

$$|\partial_\alpha z|_{\inf}(t) = \inf_{\alpha\in[0,1]} |\partial_\alpha z(\alpha,t)| > 0,$$

$$|\partial_\alpha z|_\gamma(t) = \sup_{\alpha\neq\beta} \frac{|\partial_\alpha z(\alpha,t) - \partial_\alpha z(\beta,t)|}{|\alpha-\beta|^\gamma} < \infty,$$

for all $t \in [0,T]$. Take the cut-off distance

$$\delta = \min_{\tau\in[0,t]} \left(\frac{|\partial_\alpha z|_{\inf}(t)}{|\partial_\alpha z|_\gamma(t)}\right)^{1/\gamma} > 0,$$

which is a fixed positive quantity by the previous result. Denote

$$d(x,t) := d(x,\partial D(t))$$

the distance from x to $\partial D(t)$ and write

$$\partial_1\omega_3(x,t) = \mathrm{pv}\int_0^t\int_{\mathbb{R}^2} \frac{1}{8\pi(t-\tau)^2}\left(\frac{(x_1-y_1)^2}{2(t-\tau)} - 1\right) e^{-|x-y|^2/4(t-\tau)}\theta(y,\tau)dyd\tau = I_1 + I_2, \tag{2.25}$$

where pv denotes principal value defined as in (2.18) and the splitting consists in

$$I_1 = \int_0^t \int_{D(t-\tau)\cap|y|\geq\delta} \frac{1}{8\pi\tau^2}\left(\frac{y_1^2}{2\tau} - 1\right) e^{-|y|^2/4\tau} dyd\tau,$$

$$I_2 = \mathrm{pv}\int_0^t \int_{D(t-\tau)\cap|y|\leq\delta} \frac{1}{8\pi\tau^2}\left(\frac{y_1^2}{2\tau} - 1\right) e^{-|y|^2/4\tau} dyd\tau. \tag{2.26}$$

The bound for I_1 follows from the fact that δ is a fixed positive quantity:

$$|I_1| \leq \int_0^t\int_\delta^\infty\int_0^{2\pi} \left(\frac{r^3\cos^2\alpha}{2\tau^3} + \frac{r}{\tau^2}\right)\frac{e^{-r^2/4\tau}}{8\pi} d\alpha dr d\tau = \left(\frac{4t}{\delta^2} + \frac{1}{2}\right) e^{-\delta^2/4t}. \tag{2.27}$$

Now we deal with I_2. First, if $\delta < d(x,\tau)$ for all $\tau \in [0,t]$, then cancellation (2.20) yields

$$|I_2| \leq \frac{1}{2} e^{-\delta^2/4t}. \tag{2.28}$$

Let's consider then the case in which the boundary of the patch is close to x, $0 < d(x,\tau) \leq \delta$. Since the velocity of the patch is finite almost everywhere in time and space ($u \in L^\infty(0,T;L^\infty)$), then the distance $d(x,\tau)$ is a Lipschitz function in time in $[0,T]$. In fact,

$$d(x,\tau) = \|x - x^*(\tau)\|, \ \ x^*(\tau) = \underset{y(\tau)\in\partial D(\tau)}{\arg\min} \|x - y(\tau)\|,$$

$$d(x,\tau) = d(x,0) + \int_0^\tau u(x^*(\tau')) \cdot n(x^*(\tau'))d\tau',$$

so

$$|d(x,\tau) - d(x,0)| \leq U\tau, \quad |d(x,t) - d(x,t-\tau)| \leq U\tau, \tag{2.29}$$

where $U = \|u\|_{L_T^\infty(L^\infty)}$. We will decompose the space domain $D(\tau) \cap |x - y| \leq \delta$ as follows:

$$\begin{aligned} S_r(x,\tau) &= \{z \in \mathbb{R}^2 : |z| = 1, x + rz \in D(\tau)\}, \\ \Sigma(x,\tau) &= \{z \in \mathbb{R}^2 : |z| = 1, \nabla\varphi(x^*(\tau)) \cdot z \geq 0)\}, \\ R_r(x,\tau) &= (S_r(x,\tau)\backslash\Sigma(x,\tau)) \cup (\Sigma(x,\tau)\backslash S_r(x,\tau)), \end{aligned} \tag{2.30}$$

and use for each τ the *Geometric Lemma* in [11]

$$|R_r(x,\tau)| \leq 2\pi\left((1+2^\gamma)\frac{d(x,\tau)}{r} + 2^\gamma\frac{r^\gamma}{\delta^\gamma}\right), \tag{2.31}$$

valid for all $\tau \in [0,T]$. From (2.26) we get

$$\begin{aligned} |I_2| \leq J_1 + J_2 + J_3, \quad \text{where} \quad J_1 &= \left|\mathrm{pv}\int_0^t \int_0^{d(x,t-\tau)} \frac{r}{4\tau^2}\left(\frac{r^2}{4\tau} - 1\right)e^{-r^2/4\tau}drd\tau\right|, \\ J_2 &= \left|\int_0^t \int_{d(x,t-\tau)}^\delta \int_{\Sigma(x,t-\tau)} \frac{r}{8\pi\tau^2}\left(\frac{r^2\cos^2\alpha}{2\tau} - 1\right)e^{-r^2/4\tau}d\alpha drd\tau\right|, \\ J_3 &= \left|\int_0^t \int_{d(x,t-\tau)}^\delta \int_{R_r(x,t-\tau)} \frac{r}{8\pi\tau^2}\left(\frac{r^2\cos^2\alpha}{2\tau} - 1\right)e^{-r^2/4\tau}d\alpha drd\tau\right|. \end{aligned} \tag{2.32}$$

We first notice that (2.29) gives us the bound

$$\begin{aligned} d(x,t-\tau)^2 e^{-d(x,t-\tau)^2/4\tau} &\leq 2(d(x,t)^2 + U^2\tau^2)e^{-d(x,t)^2/4\tau + d(x,t)U/2} \\ &\leq c(U,\delta)(d(x,t)^2 + \tau^2)e^{-d(x,t)^2/4\tau}, \end{aligned}$$

so that in J_1 the time-space cancellation in (2.20) yields the following

$$\begin{aligned} J_1 &\leq \int_0^t \frac{d(x,t-\tau)^2}{8\tau^2}e^{-d(x,t-\tau)^2/4\tau}d\tau \leq c\,(U,\delta)\int_0^t \left(\frac{d(x,t)^2}{\tau^2} + 1\right)e^{-d(x,t)^2/4\tau}d\tau \\ &\leq c\,(U,\delta)\,(1+t). \end{aligned} \tag{2.33}$$

By the parity of the kernel, J_2 can be estimated in a similar way to J_1:

$$J_2 = \left|\frac{1}{2}\int_0^t \left(\frac{-\delta^2}{8\tau^2}e^{-\delta^2/4\tau} + \frac{d(x,t-\tau)^2}{8\tau^2}e^{-d(x,t-\tau)^2/4\tau}\right)d\tau\right| \leq c\,(U,\delta)\,(1+t). \tag{2.34}$$

Finally, to bound J_3 we use the geometric lemma (2.31) and again (2.29)

$$\begin{aligned} J_3 &\leq \int_0^t \int_{d(x,t-\tau)}^{\delta} \frac{r}{8\pi\tau^2}\left(\frac{r^2}{2\tau}+1\right)|R_r(x,t-\tau)|e^{-r^2/4\tau}drd\tau \\ &\leq 3\int_0^t \int_{d(x,t-\tau)}^{\delta} \frac{r}{8\pi\tau^2}|R_r(x,t-\tau)|e^{-r^2/8\tau}drd\tau \leq L_1+L_2, \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} L_1 &= 3\frac{1+2^\gamma}{4}\int_0^t \int_{d(x,t-\tau)}^{\delta} \frac{d(x,t-\tau)}{\tau^2}e^{-r^2/8\tau}drd\tau, \\ L_2 &= \frac{3}{4}\frac{2^\gamma}{\delta^\gamma}\int_0^t \int_{d(x,t-\tau)}^{\delta} \frac{r^{1+\gamma}}{\tau^2}e^{-r^2/8\tau}drd\tau \leq 3\frac{2^\gamma}{\delta^\gamma}\int_\epsilon^\delta \frac{2}{r^{1-\gamma}}dr \leq 6\frac{2^\gamma}{\gamma}. \end{aligned} \tag{2.36}$$

By (2.29), we can find the bound for L_1

$$\begin{aligned} L_1 \leq & \int_0^t \int_{d(x,t)-U\tau}^{\delta} \frac{d(x,t)+U\tau}{\tau^2}e^{-r^2/8\tau}drd\tau = \int_0^t \int_{d(x,t)-U\tau}^{\delta} \frac{U}{\tau}e^{-r^2/8\tau}drd\tau \\ & + \int_0^t \int_{d(x,t)-U\tau}^{d(x,t)} \frac{d(x,t)}{\tau^2}e^{-r^2/8\tau}drd\tau + \int_0^t \int_{d(x,t)}^{\delta} \frac{d(x,t)}{\tau^2}e^{-r^2/8\tau}drd\tau \\ \leq & \int_0^t \frac{\sqrt{8}U}{\sqrt{\tau}}\int_{-\infty}^{\infty} e^{-y^2}dyd\tau + \int_0^t \frac{Ud(x,t)}{\tau}e^{-d(x,t)^2/8\tau}d\tau + \int_{d(x,t)}^{\delta} \frac{d(x,t)}{r^2}e^{-r^2/8t}dr \\ \leq & \, c\,(U,\delta)\,(1+t^{1/2}), \end{aligned}$$

so we conclude that

$$J_3 \leq c\,(U,\delta,\gamma)\,(1+t^{1/2}). \tag{2.37}$$

From the splitting (2.32) and the bounds (2.33), (2.34) and (2.37), it follows that

$$|I_2| \leq c\,(U,\delta,\gamma)\,(1+t). \tag{2.38}$$

Thus, from (2.25) and the bounds (2.27), (2.28) and (2.38), we conclude that

$$|\partial_1\omega_3(x,t)| \leq c\,(U,\delta,\gamma)\,(1+t),$$

and therefore

$$\|\partial_1\omega_3\|_{L^1_T(L^\infty)} \leq c\,(U,\delta,\gamma)\,(T+T^2). \tag{2.39}$$

Note that for $\partial_2\omega_3$ the proof above reduces to bound the term J_3 in (2.32), as it has zero mean on half circles. If we write the corresponding kernels (2.19) in polar coordinates

$$|\partial_1^2 K(r,\alpha,t)| \leq \frac{r^2}{8\pi t^3}e^{-r^2/4t} + \frac{1}{8\pi t^2}e^{-r^2/4t}, \qquad |\partial_1\partial_2 K(r,\alpha,t)| \leq \frac{r^2}{16\pi t^3}e^{-r^2/4t},$$

one can see that the bound (2.39) is also valid for $\partial_2\omega_3$. □

We proceed now to bound the second derivatives of the velocity. We want to show that $R_iR_j\partial_k\omega_3 \in L^1(0,T;L^\infty)$. The operators K_{ijk} involved can be decomposed in two parts, one with zero mean on half circles and another with the cancellation (2.20), as shown in (2.24). Rewrite for example the kernel $K_{112} = K^o_{112} + K^*_{112}$ in (2.22) as

$$K^*_{112}(r,\alpha) = \frac{1}{\pi(4t)^2}e^{-r^2/4t}\left(2 - \frac{4r^2\cos^2\alpha\sin^2\alpha}{4t} - 12\cos^2\alpha\sin^2\alpha\right),$$

$$K^o_{112}(r,\alpha) = \frac{1}{\pi r}\left(3 - 24\cos^2\alpha\sin^2\alpha\right)G(r,t),$$

where $G(r,t)$ is given by (2.23).

Using the same decomposition (2.30), we notice from (2.24) that the part corresponding to K^*_{112} can be estimated in the same way as we did with $\partial_1^2 K$ obtaining the same bound (up to a constant). Following the splittings (2.25) and (2.32), as K^o_{112} has zero mean on half circles, we only need to estimate its contribution due to R_r:

$$\begin{aligned} J_4 &= \left|\int_0^t\int_{d(x,t-\tau)}^{\delta}\int_{R_r(x,t-\tau)} rK^o_{112}(r,\alpha)d\alpha dr d\tau\right| \\ &\le \int_0^t\int_{d(x,t-\tau)}^{\delta}\int_0^{2\pi}\frac{1}{\pi}|3-24\cos^2\alpha\sin^2\alpha|\,|G(r,\tau)|\,|R_r(x,t-\tau)|\,d\alpha dr d\tau. \end{aligned}$$

Using (2.31) and the fact that $G(r,\tau)\ge 0$ $\forall r,\tau\ge 0$, one gets

$$J_4 \le 54\int_0^t\int_{d(x,t-\tau)}^{\delta} G(r,\tau)\left((1+2^\gamma)\frac{d(x,t-\tau)}{r} + 2^\gamma\frac{r^\gamma}{\delta^\gamma}\right)dr d\tau \le 54(L_7+L_8), \tag{2.40}$$

where

$$L_7 = 3\int_0^t\int_{d(x,t-\tau)}^{\delta} G(r,\tau)\frac{d(x,t-\tau)}{r}dr d\tau, \qquad L_8 = \frac{2^\gamma}{\delta^\gamma}\int_0^t\int_{d(x,t-\tau)}^{\delta} G(r,\tau)r^\gamma dr d\tau.$$

In the term L_8 the singularity has been removed so we can integrate it directly

$$L_8 = \frac{2^\gamma}{\delta^\gamma}\int_0^\delta t\frac{1-e^{-r^2/4t}}{r^{3-\gamma}}dr \le \frac{2^\gamma}{4\gamma}. \tag{2.41}$$

To deal with L_7 we write it as follows:

$$L_7 = 3\int_0^t\left(\frac{d}{d\tau}\tau\right)\int_{d(x,t-\tau)}^{\delta}\frac{d(x,t-\tau)}{r}G(r,\tau)dr d\tau,$$

and we integrate by parts in time

$$\begin{aligned} L_7 &= -3\int_0^t \tau\frac{d}{d\tau}\left(\int_{d(x,t-\tau)}^{\delta}\frac{d(x,t-\tau)}{r}G(r,\tau)dr\right)d\tau + \left[\tau\int_{d(x,t-\tau)}^{\delta}\frac{d(x,t-\tau)}{r}G(r,\tau)dr\right]_0^t \\ &= P_1 + P_2. \end{aligned}$$

The second term can be bounded by noticing $\int \tau G(r,\tau)dr = \dfrac{e^{-r^2/4\tau}-1}{2r^2/\tau}$, so that

$$P_2 \le \sup_{\tau\in[0,t]} \tau \int_{d(x,t-\tau)}^{\delta} \frac{d(x,t-\tau)}{r} G(r,t)dr \le \sup_{\tau\in[0,t]} \int_{d(x,t-\tau)}^{\delta} \tau G(r,t)dr \le \frac{1}{4}.$$

Now we take the time derivative in P_1 to find

$$P_1 \le Q_1 + Q_2 + Q_3, \tag{2.42}$$

where

$$\begin{aligned} Q_1 &= 3\int_0^t \int_{d(x,t-\tau)}^{\delta} U\frac{\tau G(r,\tau)}{r} dr d\tau, \\ Q_2 &= 3\int_0^t \int_{d(x,t-\tau)}^{\delta} \frac{d(x,t-\tau)}{\tau^2} e^{-r^2/4\tau} dr d\tau, \\ Q_3 &= 3\int_0^t \tau G(d(x,t-\tau),\tau)d\tau. \end{aligned}$$

For the first one there is enough cancellation to integrate:

$$Q_1 \le c\int_0^t \left(\frac{1}{\sqrt{\tau}} \int_0^{\delta/2\sqrt{\tau}} e^{-y^2}dy + \frac{1}{\delta} e^{-\delta^2/4\tau} - 4\tau \frac{1-e^{-\delta^2/4\tau}}{\delta^3} \right) d\tau \le c\,(\delta)\,(t^{1/2}+t).$$

The second one has already been treated in L_1 (2.36). Finally, the last term follows from the fact that

$$G(r,\tau) \le c\,\tau^{-3/2}.$$

Therefore, it is clear that

$$L_7 \le c\,(U,\delta)\,(1+t)$$

and thus, from (2.41) and (2.40),

$$J_4 \le c\,(U,\delta,\gamma)\,(1+t). \tag{2.43}$$

One can see from their expression in (2.22) that all the terms appearing in the different kernels have already been studied as part of K^*_{112} or K^o_{112}. So finally, combining the bound (2.39) (corresponding in this case to the term K^*_{112}) and (2.43), from (2.17) we get the estimate

$$\|\nabla^2 v_3\|_{L^1_T(L^\infty)} \le c\,(U,\delta,\gamma)\,(T+T^2).$$

□

Remark 2.3.2. *From the bounds above, we notice that we get indeed $v_3 \in L^\infty(0,T;W^{2,\infty})$. Using (2.14) to interpolate, we find $\nabla^2 v_1 \in L^q(0,T;W^{2,\infty})$, $q \in (1,2/(2-s)) \subset (1,4/3)$. Taking into account (2.15), we finally prove that $u \in L^q(0,T;W^{2,\infty})$ for $q \in (1,2/(2-s))$.*

Remark 2.3.3. *We note now that the ideas of striated regularity also show that the boundedness of the operator $\partial_i^\perp \partial_j(-\Delta)^{-1}\partial_k\partial_1(\partial_t - \Delta)_0^{-1}$ reduces to estimating $\Delta(\partial_t - \Delta)_0^{-1}\theta$. Indeed, as*

$$\widehat{\nabla^2 v_3}(\xi, t) = \int_0^t \frac{\xi_i^\perp \xi_j \xi_k \xi_1}{|\xi|^2} e^{-(t-\tau)|\xi|^2} \hat{\theta}(\xi, \tau) d\xi d\tau + \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \int_0^t \xi_k \xi_1 e^{-(t-\tau)|\xi|^2} \hat{\theta}(\xi, \tau) d\tau,$$

one can use the decomposition (Lemma 7.41, [7])

$$\xi_i \xi_j = a_{ij}(x)|\xi|^2 + \sum_l b_{ij}^l \xi_l (W(x) \cdot \xi),$$

where the function a_{ij} are bounded, b_{ij} are Hölder continuous and W is a vector field along which θ has some extra regularity (for example, the tangent vector field), so that $W \cdot \nabla\theta$ is Hölder continuous (in the case of a patch and the tangent vector field, $W \cdot \theta \equiv 0$ in the sense of distributions). Thus one can write

$$\widehat{\nabla^2 v_3} = \int_0^t a_{ij}(x)\xi_k\xi_1 e^{-(t-\tau)|\xi|^2}\hat{\theta}(\xi)d\xi + \int_0^t \sum_l b_{ij}^l \frac{\xi_k}{|\xi|^2}(W(x)\cdot\xi\hat{\theta}(\xi))d\xi + \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\xi_k\xi_1 e^{-t|\xi|^2}\theta(\xi),$$

hence applying the above decomposition again, the estimation of $\nabla^2 v_3$ in $L^1(0, T; L^\infty)$ reduces to estimate $\|\Delta(\partial_t - \Delta^{-1})\theta\|_{L^1(0,T;L^\infty)}$ and to control the striated regularity of the second term, which can be done by choosing W as the vector field tangent to the temperature patch (see Theorem 7.40 in [7] or Section 2.4).

The above ideas rely strongly on paradifferential calculus techniques. In addition, the quadratic form of the double Riesz transform is essential in the decomposition used, so we would still have to deal with $\Delta(\partial_t - \Delta)_0^{-1}$ (i.e., $\partial_k \omega_3$).

2.4 Persistence of $C^{2+\gamma}$ regularity

To get the propagation of this extra regularity for the patch we cannot proceed as before: to use the trajectories, one would need the velocity to have γ-Hölder continuous second derivatives. However, it is sufficient to control this regularity in the tangential direction.

In order to do so, we will need some results related to the linear transport and heat equations in Sobolev and Hölder spaces. First, we recall the definition of Besov spaces (see [7] for details). Consider the nonhomogeneous Littlewood-Paley decomposition:

Let $B = \{|\xi| \in \mathbb{R}^2 : |\xi| \leq 4/3\}$ and $C = \{|\xi| \in \mathbb{R}^2 : 3/4 \leq |\xi| \leq 8/3\}$, and fix two smooth radial functions χ and φ supported in B, C, respectively, and satisfying

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2. \tag{2.44}$$

The nonhomogeneous dyadic blocks are defined as $\Delta_j f = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\hat{f}(\xi))$ for $j \geq 0$ and $\Delta_{-1} f = \mathcal{F}^{-1}(\chi(\xi)\hat{f}(\xi))$. Then, the Besov space $B^\gamma_{p,q}(\mathbb{R}^2)$, $\gamma \in \mathbb{R}$, $p, q \in [1, \infty]$ is defined by

$$B^\gamma_{p,q}(\mathbb{R}^2) = \{u \in S'(\mathbb{R}^2) : \|u\|_{B^\gamma_{p,q}} = \|2^{j\gamma}\|\Delta_j u\|_{L^p}\|_{l^q(j \geq -1)} < \infty\}, \tag{2.45}$$

where $S'(\mathbb{R}^2)$ denotes the space of tempered distributions over $\mathbb{R}^2$. We recall that $H^s = B^s_{2,2}$ and $C^\gamma = B^\gamma_{\infty,\infty}$ for $s \in \mathbb{R}$, $\gamma \in (0,1)$.

Proposition 2.4.1. *Let $s > 0$, $r \in (1,\infty]$. Then, the following estimates hold*

$$\|\partial_i\partial_k(\partial_t-\Delta)_0^{-1}f\|_{L^r_T(C^\gamma)} \le c\|f\|_{L^r_T(C^\gamma)}, \tag{2.46}$$

$$\|\partial_i\partial_k(\partial_t-\Delta)_0^{-1}f\|_{L^1_T(C^\gamma)} \le c\|f\|_{L^1_T(C^{\gamma+s})}, \tag{2.47}$$

$$\|\partial_i\partial_k e^{t\Delta}u_0\|_{L^1_T(C^\gamma)} \le c\|u_0\|_{C^{\gamma+s}}. \tag{2.48}$$

Furthermore, there exists $u_0 \in C^\gamma$ for which $\partial_i\partial_k e^{t\Delta}u_0 \notin L^1_T(C^\gamma)$.

Proof: The proof of (2.46) can be found in [82]. The proof of (2.47) follows from Bernstein inequalities and the decay of the heat kernel:

$$\begin{aligned}\|\partial_i\partial_k(\partial_t-\Delta)_0^{-1}f\|_{L^1_T(C^\gamma)} &\le c\int_0^T \sup_{j\ge -1} 2^{j(\gamma+s)}2^{j(2-s)}\int_0^t e^{-c(t-\tau)2^{2j}}\|\Delta_j f\|_{L^\infty}d\tau dt\\ &\le c\int_0^T\int_0^t \frac{c}{(t-\tau)^{1-s/2}}\sup_{j\ge -1}2^{j(\gamma+s)}\|\Delta_j f\|_{L^\infty}d\tau dt \le c(T)\|f\|_{L^1_T(B^{\gamma+s}_{\infty,\infty})}.\end{aligned}$$

We get (2.48) as before

$$\begin{aligned}\|\partial_i\partial_k e^{t\Delta}u_0\|_{L^1_T(C^\gamma)} &\le c\int_0^T \sup_{j\ge -1} 2^{j(\gamma+s)}2^{j(2-s)}e^{-ct2^{2j}}\|\Delta_j u_0\|_{L^\infty}dt\\ &\le c\int_0^T \frac{\|u_0\|_{C^{\gamma+s}}}{t^{1-s/2}}dt \le c(T)\|u_0\|_{C^{\gamma+s}}.\end{aligned}$$

To prove the last statement we proceed as follows

$$\|\partial_i\partial_k e^{t\Delta}u_0\|_{C^\gamma} \ge c\sup_{j\ge 0}2^{j\gamma}2^{2j}\|\Delta_j(e^{t\Delta}u_0)\|_{L^\infty} \ge c\sup_{j\ge 0}2^{j\gamma}2^{2j}e^{-ct2^{2j}}\|\Delta_j u_0\|_{L^\infty},$$

therefore

$$\|\partial_i\partial_k e^{t\Delta}\|_{L^1_T(C^\gamma)} \ge c\left\|\sup_{j\ge 0}\left(2^{2j}e^{-ct2^{2j}}\right)\inf_{j\ge 0}\left(2^{j\gamma}\|\Delta_j u_0\|_{L^\infty}\right)\right\|_{L^1_T} = c\left\|\frac{1}{t}\inf_{j\ge 0}\left(2^{j\gamma}\|\Delta_j u_0\|_{L^\infty}\right)\right\|_{L^1_T}.$$

Thus one only needs to find $u_0 \in C^\gamma$ such that $\inf_{j\ge 0}\left(2^{j\gamma}\|\Delta_j u_0\|_{L^\infty}\right) > 0$. It is not difficult to check that the function define by $\hat{u}_0(\xi) = \frac{1}{|\xi|^{2+\gamma}}(1-\chi(\xi))$ satisfies the condition. □

Now we adapt these estimates to negative Hölder spaces.

Proposition 2.4.2. *Let $g(x,t) = \nabla\cdot f(x,t)$, $r \in (1,\infty]$, then we have the following estimates*

$$\left|\left|(\partial_t-\Delta)_0^{-1}g\right|\right|_{L^r_T(C^\gamma)} \le c\left(\|g\|_{L^r_T(B^{-2+\gamma}_{\infty,\infty})} + \|f\|_{L^r_T(L^1)}\right), \tag{2.49}$$

$$\left|\left|(\partial_t-\Delta)_0^{-1}g\right|\right|_{L^1_T(C^\gamma)} \le c\left(\|g\|_{L^1_T(B^{-2+\gamma+s}_{\infty,\infty})} + \|f\|_{L^1_T(L^1)}\right), \tag{2.50}$$

$$\left|\left|e^{t\Delta}g\right|\right|_{L^1_T(C^\gamma)} \le c\left(\|g\|_{B^{-2+\gamma+s}_{\infty,\infty}} + \|f\|_{L^1}\right). \tag{2.51}$$

Proof: First, if we denote by χ the first dyadic block in the Littlewood-Paley decomposition (2.44), we notice that

$$g \in B^{-2+\gamma}_{\infty,\infty} \Leftrightarrow \exists h \in C^\gamma \text{ such that } g = \Delta h.$$

Indeed,

$$\begin{aligned}\|h\|_{C^\gamma} &= 2^{-\gamma}\left\|\mathcal{F}^{-1}\left(\chi(\xi)\frac{\xi}{|\xi|^2}\cdot \hat{f}(\xi)\right)\right\|_{L^\infty} + \sup_{j\geq 0} 2^{j\gamma}\|\Delta_j(\Delta^{-1}g)\|_{L^\infty}\\ &\leq 2^{-\gamma}\left\|\mathcal{F}^{-1}\left(\chi(\xi)\frac{\xi}{|\xi|^2}\cdot \hat{f}(\xi)\right)\right\|_{L^\infty} + \|g\|_{B^{-2+\gamma}_{\infty,\infty}}.\end{aligned} \tag{2.52}$$

Note that if $f \in L^1$,

$$\left\|\mathcal{F}^{-1}\left(\chi(\xi)\frac{\xi}{|\xi|^2}\cdot \hat{f}(\xi)\right)\right\|_{L^\infty} \leq c\|f\|_{L^1},$$

so it holds that

$$\|h\|_{C^\gamma} \leq c\|f\|_{L^1} + \|g\|_{B^{-2+\gamma}_{\infty,\infty}}.$$

Applying the classic estimates (2.46), (2.47) and (2.48) to $(\partial_t - \Delta)_0^{-1} g = \Delta(\partial_t - \Delta)_0^{-1} h$ and $e^{t\Delta} g = \Delta e^{t\Delta} h$, (2.49), (2.50) and (2.51) follow.

□

We are now prepared to state the propagation of regularity for $C^{2+\gamma}$ interfaces:

Theorem 2.4.3. *Assume $\gamma \in (0,1)$. Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in C^{2+\gamma}$, $u_0 \in H^{1+\gamma+s}$, $s \in (0, 1-\gamma)$ a divergence-free vector field and $\theta_0 = 1_{D_0}$. Then,*

$$\theta(x,t) = 1_{D(t)}(x) \text{ with } \partial D \in L^\infty(0,T; C^{2+\gamma}),$$

where $D(t) = X(D_0, t)$, and there exists a unique global solution (u, θ) of (2.1)-(2.3) such that

$$u \in L^\infty(0,T; H^{1+\gamma+s}) \cap L^2(0,T; H^{2+\mu}) \cap L^p(0,T; H^{2+\delta}) \cap L^q(0,T; W^{2,\infty}),$$

for any $T > 0$, $\mu < \frac{1}{2}$, $\mu \leq \gamma + s$, $\delta < \frac{1}{2}$, $1 \leq p < 2/(1-(\gamma+s-\delta))$, $1 \leq q < 2/(2-(\gamma+s))$.

Proof: As $u_0 \in H^{1+\gamma+s}$ and $\theta_0 \in H^\delta$, for any $\delta \in (0, 1/2)$, from [79] one gets that $u \in L^\infty(0,T; H^{1+\mu}) \cap L^2(0,T; H^{2+\mu})$ with $\mu < 1/2$, $\mu \leq \gamma + s$. Using the splitting (2.12) to bootstrap, for the initial data we find that

$$\omega_0 \in H^{\gamma+s} \Longrightarrow \omega_1 \in L^\infty(0,T; H^{\gamma+s}) \cap L^1(0,T; H^{2+\gamma+\tilde{s}}), \quad \tilde{s} \in (0,s),$$

while for the convection term it holds that

$$\left.\begin{array}{l} u \in L^\infty(0,T; H^{1+\mu}) \\ \omega \in L^\infty(0,T; H^\mu) \\ \omega \in L^2(0,T; H^{1+\mu}) \end{array}\right\} \Longrightarrow \left.\begin{array}{l} u\omega \in L^\infty(0,T; H^\mu) \\ u\omega \in L^2(0,T; H^{1+\mu}) \end{array}\right\} \Rightarrow \omega_2 \in L^\infty(0,T; H^{1+\mu}) \cap L^2(0,T; H^{2+\mu}).$$

The third term remains as before $\omega_3 \in L^\infty(0,T; H^{1+\delta})$. Hence, $\omega \in L^\infty(0,T; H^{\gamma+s})$ and therefore $u \in L^\infty(0,T; H^{1+\gamma+s})$. By interpolation, $v_1 \in L^p(0,T; H^{2+\delta})$, where if $\gamma + s \geq 1/2$

then $p = \frac{2}{1-(\gamma+s-\delta)}$ and if $\gamma + s \leq 1/2$ then $p = 2\frac{1-(s-\tilde{s})}{1-(\gamma+s-\delta)-(s-\tilde{s})}$, so $u \in L^p(0,T;H^{2+\delta})$, where $1 \leq p < \frac{2}{1-(\gamma+s-\delta)}$.

Let's consider the level-set characterization of the patch:

$$D_0 = \{x \in \mathbb{R}^2 : \varphi_0(x) > 0\},$$

which for time t, $D(t) = X(t, D_0)$, is given by the function $\varphi(t,\cdot)$:

$$\partial_t\varphi + u \cdot \nabla\varphi = 0, \qquad \varphi(x,0) = \varphi_0(x).$$

The vector field $W(t) = \nabla^\perp\varphi(t)$ is then tangent to $\partial D(t)$ and its evolution is given by

$$\partial_t W + u \cdot \nabla W = W \cdot \nabla u, \quad W(0) = \nabla^\perp\varphi_0. \tag{2.53}$$

In order to control the $C^{2+\gamma}$ regularity of $\partial D(t)$ one just need to show that ∇W remains in C^γ. By differentiating (2.53) one obtains

$$\partial_t \nabla W + u \cdot \nabla(\nabla W) = W \cdot \nabla(\nabla u) + \nabla W \cdot \nabla u + \nabla u \cdot \nabla W. \tag{2.54}$$

It is clear that we can choose φ_0 such that

$$W_0 \in L^2 \cap L^\infty, \quad \nabla W_0 \in L^2 \cap L^\infty. \tag{2.55}$$

Then, from (2.53) and (2.54) we deduce that

$$W \in L^\infty(0,T;L^2 \cap L^\infty), \quad \nabla W \in L^\infty(0,T;L^2 \cap L^\infty), \tag{2.56}$$

since we know that $\nabla u \in L^1(0,T;L^\infty)$, $\nabla^2 u \in L^1(0,T;L^\infty)$.

As u is Lipschitz, the following estimate holds for all $t \in [0,T]$:

$$\|\nabla W\|_{C^\gamma}(t) \leq \|\nabla W_0\|_{C^\gamma} e^{\gamma\int_0^t \|\nabla u\|_{L^\infty} d\tau} + e^{\gamma\int_0^t \|\nabla u\|_{L^\infty} d\tau} \int_0^t \left(\|W \cdot \nabla^2 u\|_{C^\gamma} + 2\|\nabla W\|_{C^\gamma}\|\nabla u\|_{C^\gamma}\right) d\tau.$$

From this and previous estimates we get that

$$\|\nabla W\|_{C^\gamma}(t) \leq c_1(T) + c_2(T)\int_0^t \|W \cdot \nabla^2 u\|_{C^\gamma}(\tau)d\tau + c_3(T)\int_0^t \|\nabla u\|_{C^\gamma}(\tau)\|\nabla W\|_{C^\gamma}(\tau)d\tau. \tag{2.57}$$

We need to decompose the term $W \cdot \nabla^2 u$ above to benefit from the extra cancellation in the tangential direction. We will use the following lemma:

Lemma 2.4.4. *For any $\gamma \in (0,1)$, there exists a constant C such that the following estimate holds true:*

$$\|W \cdot \nabla^2 u\|_{L^1_t(C^\gamma)} \leq C(\gamma)\left(\int_0^t \|W\|_{C^\gamma}\left(\|\nabla\omega\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty}\right) dt + \int_0^t \|\nabla \cdot (W\omega)\|_{C^\gamma} dt\right). \tag{2.58}$$

Proof of Lemma 2.4.4: We decompose as follows

$$W(x)\cdot\nabla(\partial_j u_i)(x)=F(x)+\frac{1}{2\pi}\mathrm{pv}\int\frac{\sigma_{ij}(x-y)}{|x-y|^2}W(y)\cdot\nabla\omega(y)dy+\frac{W(x)\cdot\nabla\omega(x)}{2}\begin{bmatrix}0&-1\\1&0\end{bmatrix},$$

where

$$F(x)=\frac{1}{2\pi}\mathrm{pv}\int\frac{\sigma_{ij}(x-y)}{|x-y|^2}(W(x)-W(y))\cdot\nabla\omega(y)dy,\quad \sigma_{ij}(x)=\frac{1}{|x|^2}\begin{bmatrix}2x_1x_2 & x_2^2-x_1^2\\ x_2^2-x_1^2 & -2x_1x_2\end{bmatrix}.$$

As the vector field W is divergence-free, it yields that

$$\|W\cdot\nabla^2u\|_{C^\gamma}\leq||F||_{C^\gamma}+c\|\nabla\cdot(W\omega)\|_{C^\gamma}.$$

By the Lemma in Appendix of [11],

$$\|F\|_{C^\gamma}\leq C(\gamma)\|W\|_{C^\gamma}(\|\nabla\omega\|_{L^\infty}+\|\nabla^2u\|_{L^\infty}),$$

and the result follows. □

Hence, as $u\in L^1(0,T;W^{2,\infty})$ from Theorem 2.3.1, (2.58) yields

$$\int_0^t\|W\cdot\nabla^2u\|_{C^\gamma}(\tau)d\tau\leq c(T)+c(\gamma)\int_0^t\|\nabla\cdot(W\omega)\|_{C^\gamma}d\tau.$$

Going back to (2.57),

$$\|\nabla W\|_{C^\gamma}(t)\leq c_1(T)+c_2(T)\int_0^t\|\nabla\cdot(W\omega)\|_{C^\gamma}(\tau)d\tau+c_3(T)\int_0^t\|\nabla u\|_{C^\gamma}\|\nabla W\|_{C^\gamma}(\tau)d\tau. \quad (2.59)$$

So the problem is reduced to control the tangential derivatives of the vorticity in $L^1(0,t;C^\gamma)$. We will use the properties of the heat kernel in Hölder spaces applied to the equation satisfied by $\nabla\cdot(W\omega)$:

$$\partial_t\nabla\cdot(W\omega)+u\cdot\nabla(\nabla\cdot(W\omega))-\Delta\nabla\cdot(W\omega)=\nabla\cdot(W\partial_1\theta)+\nabla\cdot(W\Delta\omega-\Delta(W\omega)),$$
$$\nabla\cdot(W\omega)|_{t=0}=\nabla\cdot(W_0\omega_0).$$

Notice that since W is a divergence-free vector field tangent to the patch θ, it is possible to find

$$\nabla\cdot(W\partial_1\theta)=\nabla\cdot(\partial_1(W\theta)-\theta\partial_1W)=-\nabla\cdot(\theta\partial_1W).$$

The solution can be decomposed as follows

$$\begin{aligned}\nabla\cdot(W\omega)(t)&=e^{t\Delta}(\nabla\cdot(W_0\omega_0))-(\partial_t-\Delta)_0^{-1}\nabla\cdot(u\nabla\cdot(W\omega))\\&\quad+(\partial_t-\Delta)_0^{-1}\nabla\cdot(W\Delta\omega-\Delta(W\omega))-(\partial_t-\Delta)_0^{-1}\nabla\cdot(\theta\partial_1W)\\&=G_1+G_2+G_3+G_4.\end{aligned} \quad (2.60)$$

We will use now some classic estimates for solutions of the heat equation in Hölder spaces to achieve the gain of two derivatives integrable in time. We can apply (2.51) to G_1 to obtain

$$\begin{aligned}\|G_1\|_{L^1_t(C^\gamma)} &\le c\left(\|\nabla\cdot(W_0\omega_0)\|_{B^{-2+\gamma+s}_{\infty,\infty}} + \|W_0\omega_0\|_{L^1}\right) \le c\left(\|W_0\omega_0\|_{B^{-1+\gamma+s}_{\infty,\infty}} + \|W_0\|_{L^2}\|\omega_0\|_{L^2}\right)\\ &= c\left(\|\nabla^\perp\cdot(W_0\otimes u_0) - u_0\cdot\nabla^\perp W_0\|_{B^{-1+\gamma+s}_{\infty,\infty}} + \|W_0\|_{L^2}\|\omega_0\|_{L^2}\right)\\ &\le c\left(\|W_0\|_{C^{\gamma+s}}\|u_0\|_{C^{\gamma+s}} + \|u_0\|_{L^\infty}\|\nabla^\perp W_0\|_{L^\infty} + \|W_0\|_{L^2}\|\omega_0\|_{L^2}\right),\end{aligned}$$

thus

$$\|G_1\|_{L^1_t(C^\gamma)} \le c(t). \tag{2.61}$$

Estimate (2.49) yields

$$\begin{aligned}\|G_2\|_{L^1_t(C^\gamma)} &\le c\left(\|\nabla\cdot(u\nabla\cdot(W\omega))\|_{L^r_t(B^{-2+\gamma}_{\infty,\infty})} + \|u\nabla\cdot(W\omega)\|_{L^r_t(L^1)}\right)\\ &\le c\left(\|\nabla\cdot(u\otimes W\omega) - \omega W\cdot\nabla u\|_{L^r_t(B^{-1+\gamma}_{\infty,\infty})} + \|\|u\|_{L^2}\|W\|_{L^\infty}\|\nabla\omega\|_{L^2}\|_{L^r_t}\right)\\ &\le c\Big(\|W\|_{L^\infty_t(C^\gamma)}\|u\|_{L^\infty_t(C^\gamma)}\|\omega\|_{L^r_t(C^\gamma)} + \|W\|_{L^\infty_t(L^\infty)}\|\omega\|_{L^{2r}_t(L^\infty)}\|\nabla u\|_{L^{2r}_t(L^\infty)}\\ &\quad + \|W\|_{L^\infty_t(L^\infty)}\|u\|_{L^\infty_t(L^2)}\|\nabla\omega\|_{L^r_t(L^2)}\Big).\end{aligned}$$

Now, as in Remark 2.3.2, since $v_1 \in L^1(0,T;H^{3+\gamma+\tilde{s}})$ and $v_1 \in L^2(0,T;H^{2+\gamma+s})$, by interpolation we obtain

$$u \in L^q(0,t;W^{2,\infty}), \qquad \text{with } q \in \left[1, \frac{2}{2-(\gamma+s)}\right),$$

and similarly for any $0<\epsilon<\mu$

$$\omega \in L^\infty(0,T;H^{\gamma+s}) \cap L^2(0,T;H^{1+\mu}) \Longrightarrow \omega \in L^{2r}(0,T;H^{1+\epsilon}),$$

with

$$1 < r \le \frac{1-\gamma-s+\mu}{1-\gamma-s+\epsilon} < 1 + \frac{\mu}{1-(\gamma+s)}.$$

Therefore, by choosing $1 < r < \min\{2/(2-(\gamma+s)), 1+\mu/(1-(\gamma+s))\}$ we get that

$$\|G_2\|_{L^1_t(C^\gamma)} \le c(t). \tag{2.62}$$

In the term G_3 we need to use the following commutator:

$$W\Delta\omega - \Delta(W\omega) = -\omega\Delta W - 2\nabla W\cdot\nabla\omega = -\nabla\cdot(\omega\nabla W) - \nabla W\cdot\nabla\omega,$$

$$G_3 = -\int_0^t e^{(t-\tau)\Delta}\nabla\cdot(\nabla\cdot(\omega\nabla W))d\tau - \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(\nabla W\cdot\nabla\omega)d\tau = G_{31} + G_{32}.$$

Notice that in G_{31} we find two derivatives applied to $\omega\nabla W$. Therefore, proceeding as in (2.52), we can apply a similar estimate to (2.49) in order to find that

$$\begin{aligned}\|G_{31}\|_{L^1_t(C^\gamma)} &\le c\left(\|\nabla\cdot(\omega\nabla W)\|_{L^r_t(B^{-1+\gamma}_{\infty,\infty})} + \|\mathcal{F}^{-1}(\chi(\xi)\widehat{\omega\nabla W})\|_{L^r_t(L^\infty)}\right)\\ &\le \left(\|\|\omega\|_{C^\gamma}\|\nabla W\|_{C^\gamma}\|_{L^r_t} + \|\nabla W\|_{L^\infty_t(L^2)}\|\omega\|_{L^r_t(L^2)}\right),\end{aligned}$$

so taking into account (2.56), we get

$$\|G_{31}\|_{L^1_t(C^\gamma)} \leq c\|\|\omega\|_{C^\gamma}\|\nabla W\|_{C^\gamma}\|_{L^r_t} + c(t). \tag{2.63}$$

For G_{32} applying (2.50) yields that

$$\begin{aligned}\|G_{32}\|_{L^1_t(C^\gamma)} &\leq c\left(\|\nabla W\cdot\nabla\omega\|_{L^1_t(B^{-1+\gamma+s}_{\infty,\infty})} + \|\nabla W\cdot\nabla\omega\|_{L^1_t(L^1)}\right)\\ &\leq c\left(\|\nabla W\|_{L^\infty(L^\infty)}\|\nabla\omega\|_{L^1_t(L^\infty)} + \|\nabla W\|_{L^\infty_t(L^\infty)}\|\nabla\omega\|_{L^1_t(L^2)}\right),\end{aligned}$$

and therefore

$$\|G_{32}\|_{L^1_t(C^\gamma)} \leq c(t). \tag{2.64}$$

Joining (2.63) and (2.64) we obtain

$$\|G_3\|_{L^1_t(C^\gamma)} \leq c(t) + c\|\|\omega\|_{C^\gamma}\|\nabla W\|_{C^\gamma}\|_{L^r_t}. \tag{2.65}$$

Finally, for the temperature term we use that $\nabla\cdot(W\theta) = W\cdot\nabla\theta \equiv 0$ in the sense of distributions:

$$\begin{aligned}\|G_4\|_{L^1_t(C^\gamma)} &\leq c\left(\|\nabla\cdot(\theta\partial_1 W)\|_{L^1_t(B^{-2+\gamma+s}_{\infty,\infty})} + \|\theta\partial_1 W\|_{L^1_t(L^1)}\right)\\ &\leq c\|\nabla W\|_{L^\infty_t(L^\infty)}\left(\|\theta\|_{L^1_t(L^\infty)} + \|\theta\|_{L^1_t(L^1)}\right),\end{aligned}$$

so similarly to the others terms it is easy to find that

$$\|G_4\|_{L^1_t(C^\gamma)} \leq c(t). \tag{2.66}$$

Combining the above bounds (2.61), (2.62), (2.65) and (2.66), it follows that

$$\int_0^t \|\nabla\cdot(W\omega)\|_{C^\gamma}(\tau)d\tau = \|G\|_{L^1_t(C^\gamma)} \leq c(t) + c\|\|\omega\|_{C^\gamma}\|\nabla W\|_{C^\gamma}\|_{L^r_t},$$

and going back to (2.59) we conclude that

$$\begin{aligned}\|\nabla W\|_{C^\gamma}(t) &\leq c(T)+c(T)\int_0^t \|\nabla u\|_{C^\gamma}(\tau)\|\nabla W\|_{C^\gamma}(\tau)d\tau + c(T)\left(\int_0^t \|\omega\|^r_{C^\gamma}(\tau)\|\nabla W\|^r_{C^\gamma}(\tau)d\tau\right)^{1/r}\\ &\leq c(T)+c(T)\left(\int_0^t \|\nabla u\|^r_{C^\gamma}(\tau)\|\nabla W\|^r_{C^\gamma}(\tau)d\tau\right)^{1/r}\\ &\leq c(T)+c(T)\left(\int_0^t \|\nabla u\|^{r_1}_{C^\gamma}\right)^{1/r_1}\left(\int_0^t \|\nabla W\|^{r_2}_{C^\gamma}\right)^{1/r_2} \leq c(T)+c(T)\left(\int_0^t \|\nabla W\|^{r_2}_{C^\gamma}\right)^{1/r_2},\end{aligned}$$

where we can choose $1 < r < r_1 < \min\{2/(2-(\gamma+s)), 1+\mu/(1-(\gamma+s))\}$ and $1/r_2 = 1/r - 1/r_1$. Therefore, raising to power r_2,

$$\|\nabla W\|^{r_2}_{C^\gamma}(t) \leq c(T) + c(T)\int_0^t \|\nabla W\|^{r_2}_{C^\gamma}(\tau)d\tau,$$

and hence applying Grönwall's inequality the result follows.

□

Remark 2.4.5. *The result also holds for $s = 0$, i.e., $u_0 \in H^{1+\gamma}$. We need to recall the definition of the* tilde spaces *introduced in [23]:*

$$\tilde{L}^\rho_t(B^\gamma_{p,q}(\mathbb{R}^2)) = \{u \in S'(0,t;\mathbb{R}^2) : \|u\|_{\tilde{L}^\rho_t(B^\gamma_{p,q})} = \|2^{j\gamma}\|\Delta_j u\|_{L^\rho_t(L^p)}\|_{l^q(j\geq -1)} < \infty\}.$$

In this spaces one can prove the following results for the heat and transport equations:

Proposition 2.4.6. *Let $f \in \tilde{L}^1_T(C^\gamma)$, $f_0 \in C^\gamma$. Then,*

$$\|\partial_i\partial_k(\partial_t - \Delta)_0^{-1} f\|_{\tilde{L}^1_T(C^\gamma)} \leq c\|f\|_{\tilde{L}^1_T(C^\gamma)} \leq c\|f\|_{L^1_T(C^\gamma)},$$
$$\|\partial_i\partial_k e^{t\Delta} f_0\|_{\tilde{L}^1_T(C^\gamma)} \leq c(T)\|f_0\|_{C^\gamma}.$$

Proof: By the definition of tilde *spaces we can now integrate first in time,*

$$\|\partial_i\partial_k(\partial_t - \Delta)_0^{-1} f\|_{\tilde{L}^1_T(C^\gamma)} = \sup_{j\geq -1} 2^{j\gamma}\|\Delta_j(\partial_i\partial_k(\partial_t - \Delta)_0^{-1} f)\|_{L^1_T(L^\infty)}$$
$$\leq c \sup_{j\geq -1} \left\|\int_0^t 2^{2j} e^{-c(t-\tau)2^{2j}} 2^{j\gamma}\|\Delta_j f\|_{L^\infty}(\tau)d\tau\right\|_{L^1_T} \leq c \sup_{j\geq -1} 2^{j\gamma}\|\Delta_j f\|_{L^1_T(L^\infty)} = c\|f\|_{\tilde{L}^1_T(C^\gamma)}.$$

The initial condition estimate follows from a similar procedure.

□

Proposition 2.4.7. *Let f be the solution to the transport equation*

$$f_t + u\cdot\nabla f = g, \quad f(0) = f_0,$$

where $\nabla u \in L^1_T(L^\infty)$, $g \in \tilde{L}^1_T(C^\gamma)$ and $f_0 \in C^\gamma$. Then,

$$\|f\|_{L^\infty_T(C^\gamma)} \leq \left(\|f_0\|_{C^\gamma} + \|g\|_{\tilde{L}^1_T(C^\gamma)}\right) e^{c\int_0^t \|\nabla u\|_{L^\infty} d\tau}.$$

The proof of Proposition 2.4.7 can be found in Theorem 3.14, *Chapter 3 [7].*

□

Now, Proposition 2.4.7 applied to (2.54) *yields that*

$$\|\nabla W\|_{L^\infty_T(C^\gamma)} \leq c(T) + c(T)\|W\cdot\nabla^2 u\|_{\tilde{L}^1_t(C^\gamma)} + c(T)\int_0^t \|\nabla W\|_{C^\gamma}\|\nabla u\|_{C^\gamma} d\tau. \qquad (2.67)$$

Lemma 2.4.4 can be written using $\tilde{L}^1_T$ instead of L^1_T, so applying Proposition 2.4.6 to the term $\nabla\cdot(W\omega)$ yields the result for $s = 0$.

2.5 Results with Hölder and Besov velocities

The results above have been presented using only Sobolev spaces for the velocity, but as commented before they can be stated using also Hölder or Besov spaces.

Corollary 2.5.1. *Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in W^{2,\infty}$, $u_0 \in C^{s_1} \cap L^p$ a divergence-free vector field, $s_1 \in (0,1)$, $p \in [1,2)$ and $\theta_0 = 1_{D_0}$ the characteristic function of D_0. Then,*

$$\theta(x,t) = 1_{D(t)}(x) \text{ and } \partial D \in L^\infty(0,T;W^{2,\infty}).$$

Moreover, there exists a unique global solution (u,θ) of (2.1)-(2.3) such that

$$u \in L^\infty(0,T;H^{s_2}) \cap L^2(0,T;H^{1+s_2}) \cap L^1(0,T;H^{2+s_2}) \cap L^\infty(0,T;C^{s_1}) \cap L^1(0,T;W^{2,\infty}),$$

for any $T > 0$, $0 \leq s_2 < s_1/2 < 1/2$.

Proof: First we notice that as $u_0 \in C^{s_1} \cap L^p$, it also holds that $u_0 \in L^{\tilde{p}} \cap H^{s_2}$ for $\tilde{p} \in [p,\infty]$, $s_2 \in [0, s_1/2)$. The *a priori* estimates are done exactly as in Theorem 2.2.1:

$$u \in L^\infty(0,T;H^{s_2}) \cap L^2(0,T;H^{1+s_2}) \cap L^1(0,T;H^{2+s_2}).$$

In addition, as $u_0 \in C^{s_1}$, $s_1 > 0$, one trivially gets $\nabla^2 v_1 \in L^1(0,T;L^\infty)$. To show that $u \in L^1(0,T;W^{2,\infty})$, we follow the same structure as in Theorem 2.3.1: use the splitting (2.12) and treat each one of the terms $\nabla^2 v_1$, $\nabla^2 v_2$ and $\nabla^2 v_3$, given by (2.16) separately.

To deal with the temperature term, it suffices to prove that $u \in L^\infty(0,T;L^\infty)$ to apply the steps of Theorem 2.3.1 to obtain $\nabla^2 v_3 \in L^\infty(0,T;L^\infty)$. Proposition 2.1 in [48] would yield that $u \in L^\infty(0,T;C^{s_1})$. We give a different proof that does not require the use of paradifferential calculus. We apply the Leray projector to the velocity equation to obtain

$$u_t - \Delta u = -\nabla\cdot(u\otimes u) + \nabla(-\Delta)^{-1}\left(\nabla\cdot\nabla\cdot(u\otimes u)\right) + (0,\theta) - \nabla(-\Delta)^{-1}\partial_2\theta.$$

Hence, we can write the velocity as follows:

$$\begin{gathered} u = w_1 + w_2 + w_3 + w_4 + w_5, \\ w_1 = e^{t\Delta}u_0, \quad w_2 = -(\partial_t - \Delta)_0^{-1}\nabla\cdot(u\otimes u), \quad w_3 = (\partial_t - \Delta)_0^{-1}\nabla(-\Delta)^{-1}\nabla\cdot(\nabla\cdot(u\otimes u)), \\ w_4 = (\partial_t - \Delta)_0^{-1}(0,\theta), \quad w_5 = -(\partial_t - \Delta)_0^{-1}\nabla(-\Delta)^{-1}\partial_2\theta. \end{gathered} \tag{2.68}$$

From the properties of the heat equation we obtain

$$\|w_1\|_{L^\infty_T(C^{s_1})} \leq c\|u_0\|_{C^{s_1}}, \quad \|w_4\|_{L^\infty_T(B^2_{\infty,\infty})} \leq c\|\theta\|_{L^\infty_T(L^\infty)}, \quad \|w_5\|_{L^\infty_T(B^2_{\infty,\infty})} \leq c\|\theta\|_{L^\infty_T(L^\infty)}. \tag{2.69}$$

Using the boundedness of singular integrals in Hölder spaces, w_3 can be treated as w_2. First, as $u \in L^\infty(0,T;H^{s_2}) \cap L^2(0,T;H^{1+s_2})$, by interpolation

$$u \in L^q(0,T;H^{1+\tilde{s}_2}) \hookrightarrow L^q(0,T;L^\infty), \qquad q = \frac{2}{1-(s_2-\tilde{s}_2)}.$$

Now we proceed as follows

$$\|w_2\|_{C^{s_1}}(t) \leq c\int_0^t \frac{\|u\otimes u\|_{C^{s_1}}(\tau)}{(t-\tau)^{1/2}}d\tau \leq c\int_0^t \frac{\|u\|_{C^{s_1}}\|u\|_{L^\infty}}{(t-\tau)^{1/2}}d\tau. \tag{2.70}$$

Choose $l = 2/(1+\epsilon)$ with $\epsilon \in (0,1)$ and

$$\frac{1}{r} = 1 - \frac{1}{q} - \frac{1}{l} = \frac{s_2 - \tilde{s}_2 - \epsilon}{2},$$

so applying Hölder inequality yields that

$$\|w_2\|_{C^{s_1}}(t) \leq c(T)\|u\|_{L^q_T(L^\infty)} \left(\int_0^t \|u\|^r_{C^{s_1}}(\tau)d\tau\right)^{1/r}. \tag{2.71}$$

From the decomposition (2.68), the bounds (2.69), (2.71) and recalling that $\|w_3\|_{C^{s_1}} \leq c\|w_2\|_{C^{s_1}}$, it is easy to get

$$\|u\|_{C^{s_1}}(t) \leq c(T) + c(T)\left(\int_0^t \|u\|^r_{C^{s_1}}(\tau)d\tau\right)^{1/r}.$$

Raising to the power r and applying Grönwall's inequality we conclude that

$$u \in L^\infty(0,T;C^{s_1}) \tag{2.72}$$

and therefore $u \in L^\infty(0,T;L^\infty)$.

To conclude, we need to prove that $\nabla^2 v_2 \in L^1(0,T;L^\infty)$. It suffices to show that $u\omega \in L^r(0,T;C^\delta)$ for some $r > 1, \delta > 0$. By interpolation,

$$\omega \in L^\sigma(0,T;H^{1+s_4}), \quad \sigma = \frac{2}{1+s_4} > \frac{4}{3}, \quad s_4 \in (0,s_2). \tag{2.73}$$

Thus $\omega \in L^\sigma(0,T;C^{s_4})$. From this and (2.72), $u\omega \in L^\sigma(0,T;C^\delta)$, with $\delta = \min\{s_3, s_4\} > 0$. □

Corollary 2.5.2. *Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in W^{2,\infty}$, $\theta_0 = 1_{D_0}$ and u_0 a divergence-free vector field in the Besov space $B^1_{2,1}$. Then,*

$$\partial D \in L^\infty(0,T;W^{2,\infty})$$

and there exists a unique global solution (u,θ) of (2.1)-(2.3) such that

$$u \in L^\infty(0,T;B^1_{2,1}) \cap L^2(0,T;B^2_{2,1}) \cap L^1(0,T;B^{2+\mu}_{2,1}) \cap L^1(0,T;W^{2,\infty}),$$

for any $T > 0$, $\mu < 1/2$.

Proof: One can repeat the proof of Theorem 2.3.1 to get this result. In this case, the *a priori* estimates are done using the result in [43]. Then, by splitting the vorticity equation, one can take advantage of the fact that $B^1_{2,1} \hookrightarrow L^\infty$, it is an algebra and for the heat equation with initial data in this space there is a gain of two derivatives integrable in time [24]. □

Corollary 2.5.3. *Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in C^{1+\gamma}$, $u_0 \in B^{-1+\gamma+s_1}_{\infty,\infty} \cap H^{s_2}$ a divergence-free vector field, $s_1 \in (0, 1-\gamma), s_2 \in (0,1)$ and $\theta_0 = 1_{D_0}$ the characteristic function of D_0. Then, there exists a unique global solution (u, θ) of (2.1)-(2.3) such that*

$$\theta(x,t) = 1_{D(t)}(x) \text{ and } \partial D \in L^\infty(0,T; C^{1+\gamma}).$$

Moreover,

$$u \in L^\infty(0,T; H^{s_2}) \cap L^2(0,T; H^{1+s_2}) \cap L^1(0,T; H^{2+\mu}) \cap L^\infty(0,T; B^{-1+\gamma+s_1}_{\infty,\infty}) \cap L^1(0,T; C^{1+\gamma+\tilde{s}_1}),$$

for any $T > 0$, $\mu < \min\{\frac{1}{2}, s_2\}$, $0 < \tilde{s}_1 < s_1$.

Proof: Since $u_0 \in H^{s_2}$, we get the *a priori* estimates $u \in L^\infty(0,T; H^{s_2}) \cap L^2(0,T; H^{1+s_2}) \cap L^1(0,T; H^{2+\mu})$. Now, we use the decomposition (2.68). As $u_0 \in B^{-1+\gamma+s_1}_{\infty,\infty}$, it holds that $w_1 \in L^\infty(0,T; B^{-1+\gamma+s_1}_{\infty,\infty}) \cap L^2(0,T; C^{\gamma+\sigma}) \cap L^1(0,T; C^{1+\gamma+\tilde{s}_1})$, $\tilde{s}_1 < \sigma < s_1$. Since $w_4, w_5 \in L^\infty(B^2_{\infty,\infty})$, we only need to deal with the nonlinear terms.

As $w_2 = (\partial_t - \Delta)^{-1}_0 \nabla \cdot (u \otimes u)$, it suffices to show that $u \in L^2(0,T; C^{\gamma+\sigma})$ to conclude that $w_2 \in L^1(0,T; C^{1+\gamma+\tilde{s}_1})$, $\tilde{s}_1 \in (0, s_1)$. By the estimates above, we only need to show that $w_2 \in L^2(0,T; C^{\gamma+\sigma})$. This means in turn that it suffices to show $u \otimes u \in L^2(0,T; B^{-1+\gamma+\sigma}_{\infty,\infty})$. We will show now that $u \in L^\infty(0,T; B^{-1+\gamma+s_1})$ indeed. It is clear for w_1, w_4, w_5.

Applying basic paradifferential calculus estimates (see Chapter 2 in [7]), we obtain

$$\|u \otimes u\|_{B^{-1+\gamma+s_1}_{\infty,\infty}} \leq c\|u\|_{B^{-1+\gamma+s_1}_{\infty,\infty}}(\tau)(\|u\|_{L^\infty}(\tau) + \|u\|_{H^1}). \tag{2.74}$$

Proceeding now as in (2.70),

$$\|w_2\|_{B^{-1+\gamma+s_1}_{\infty,\infty}}(t) \leq c\int_0^t \frac{\|u \otimes u\|_{B^{-1+\gamma+s_1}_{\infty,\infty}}(\tau)}{(t-\tau)^{1/2}} d\tau \leq c\int_0^t \frac{\|u\|_{B^{-1+\gamma+s_1}_{\infty,\infty}}(\tau)(\|u\|_{L^\infty}(\tau) + \|u\|_{H^1}(\tau))}{(t-\tau)^{1/2}} d\tau,$$

we find that (for the same r as in (2.71))

$$\|u\|_{B^{-1+\gamma+s_1}_{\infty,\infty}}(t) \leq c(T) + c(T)\left(\int_0^t \|u\|^r_{B^{-1+\gamma+s_1}_{\infty,\infty}}(\tau) d\tau\right)^{1/r},$$

so we conclude that $u \in L^\infty(0,T; B^{-1+\gamma+s_1}_{\infty,\infty})$. From this and the inequality (2.74) we conclude that $u \otimes u \in L^2(0,T; B^{-1+\gamma+s_1}_{\infty,\infty})$.

□

Chapter 3

Navier-Stokes density patch problem

3.1 Introduction

We consider an incompressible inhomogeneous fluid in the whole space $\mathbb{R}^2$,

$$\begin{aligned} \nabla \cdot u &= 0, \\ \rho_t + u \cdot \nabla \rho &= 0, \end{aligned} \tag{3.1}$$

driven by Navier-Stokes equations

$$\rho(u_t + u \cdot \nabla u) = \Delta u - \nabla p, \tag{3.2}$$

where the unknowns ρ, u, p represent the density, velocity field and pressure of the fluid.

In the case of positive density, the first results of existence of strong solutions for smooth initial data were proved by Ladyzhenskaya and Solonnikov [83]. When $\rho \geq 0$ is allowed, Simon [105] proved the global existence of weak solutions with finite energy. Afterwards, this result was extended to the case with variable viscosity by Lions in [88]. There, the author proposed the so-called *density patch problem*: assuming $\rho_0 = 1_{D_0}$ for some domain $D_0 \subset \mathbb{R}^2$, the question is whether or not $\rho(t) = 1_{D(t)}$ for some domain $D(t)$ with the same regularity as the initial one. Theorem 2.1 in [88] ensures that the density remains as a patch preserving its volume, but gives no information about the persistence of regularity.

Previously to this problem, the analogous question in vortex patches in Euler equations arose great interest, due to the fact that several numerical results indicated the possible formation of finite time singularities. First Chemin [22] using paradifferential calculus and later Bertozzi and Constantin [11] by a geometrical harmonic analysis approach finally solved the *vortex patch problem* proving the contrary: $C^{1+\gamma}$ vortex patches preserve their regularity in time.

On the other hand, the appearance of finite-time singularities has been proved in related scenarios. For the Muskat problem density patches have been shown to become singular in finite time [15], [16]. When vacuum is considered for Euler equations with gravity, 'splash' singularities were shown in the so-called water wave problem [17]. Later these results were extended to parabolic problems such as Muskat [20] and Navier-Stokes [18]. Different proofs of these results can be found in [40], [41]. In [56] it is shown that the presence of a second fluid precludes 'splash' singularities in Euler equations with gravity and surface tension. See [60] for a different proof applied to the Muskat problem and also [42] for the result including vorticity in the bulk.

Global-in-time regularity has been extensively studied for Navier-Stokes free boundary problems considering the continuity of the stress tensor at the free boundary (see [116] and [51] for a discussion of physical free boundary conditions). Starting from the nowadays classical local existence results [108], [8], global existence was achieved in [109], [112] for the scenario of an almost horizontal viscous fluid lying above a bottom and below vacuum. See also [72], [69], [70], [71] where the decay rate in time of the solution is studied in the previous situation in order to understand the long-time dynamics. In the two-fluid case (also known as internal waves problem in this scenario) global well-posedness and decay have been shown in [115]. See also [93] for the vanishing viscosity limit problem for the free-boundary Navier-Stokes equations.

Recently several contributions have been made in the two-fluid case without viscosity jump with low regular positive density. First, Danchin and Mucha [46], [47] showed the global well-posedness of (3.1)-(3.2) for initial densities allowing discontinuities across C^1 interfaces with a sufficiently small jump and small initial velocity in the Besov space $B^{2/p-1}_{p,1}$, $p \in [1,4)$ (see (2.45) for the definition). For densities close enough to a positive constant and initial velocity in $\dot{B}^{2/p-1}_{p,1} \cap \dot{B}^{2/p-1+\epsilon}_{p,1}$, Huang, Paicu and Zhang [80] obtained solutions with $C^{1+\epsilon}$ flow for small enough $\epsilon > 0$. Later Paicu, Zhang and Zhang [100] obtained the global-wellposedness with initial data $u_0 \in H^s$, $s \in (0,1)$ and initial positive density bounded from below and above removing the smallness conditions.

Based on these results and using paradifferential calculus and the techniques of striated regularity, Liao and Zhang have recently proved the persistence of $W^{k,p}$ regularity, $k \geq 3$, $p \in (2,4)$, for initial patches of the form

$$\rho_0 = \rho_1 1_{D_0} + \rho_2 1_{D_0^c},$$

first assuming $\rho_1, \rho_2 > 0$ close to each other [86], then for any pair of positive constants [87]. By Sobolev embedding, this means that the boundary of the patch must be at least in $C^{2+\gamma}$ for some $\gamma > 0$. Using the well-posedness result in [46], Danchin and Zhang [48] have recently obtained the propagation of $C^{1+\gamma}$ patches for small jump and small u_0 (also for large u_0 but only locally in time).

In this chapter we consider the 2D density patch problem for Navier-Stokes without any smallness condition on the initial data and without any restriction on the density jump.

First, we show that initial $C^{1+\gamma}$ density patches preserve their regularity globally in time for any $\rho_1, \rho_2 > 0$ and any $u_0 \in H^{\gamma+s}$, $s \in (0, 1-\gamma)$. We note that the cancellation in the tangential direction to the patch is not needed to propagate low regularities. Although one cannot expect to get the needed regularity for the velocity in Sobolev spaces, we will take advantage of the fact that ρ remains as a patch with Lipschitz boundary. The quasilinear character of the coupling between density and velocity makes it harder to propagate the regularity of the velocity and hence that of the patch. However, we will prove that $u \in L^1(0,T;C^{1+\gamma})$ and thus the propagation follows by the particle trajectories system

$$\begin{cases} \dfrac{dX}{dt}(x,t) = u(X(x,t),t), \\ X(x,0) = x. \end{cases}$$

Without considering regularity in the tangential direction to the density patch for the initial velocity, the initial conditions in [87] is at the level of $u_0 \in B^{1+\gamma+s}_{2,1}$ giving regularity $C^{2+\gamma}$. Analogously, in [49] the velocity is at the level of $u_0 \in B^{\gamma}_{2,1}$ giving regularity $C^{1+\gamma}$. Indeed, as in [62], from the results of maximum regularity of the linear heat equation, we deem $u_0 \in H^{\gamma+s}$ is sharp at the scale of Sobolev spaces from this approach.

This low regularity result combined with new ideas allows us to show that the curvature of patches with initial $W^{2,\infty}$ regularity remains bounded for all time. Following the particle trajectory method to preserve the regularity, the curvature is controlled once that $\nabla^2 u \in L^1(0,T;L^\infty)$. This is critical because at this level of derivatives the step function ρ appears together with the nonlinearity. So in principle one could find that $\nabla^2 u \in L^1(0,T;BMO)$. It is possible to use time weighted energy estimates, introduced in [76], [77] for the compressible model and in [100] for the incompressible case, combined with the characterization of a patch as a Sobolev multiplier to get higher regularity. In particular, to deal with the regularity of u_t the convective derivative approach in [87] can be used. Going further, the $C^{1+\gamma}$ regularity result in conjunction with the cancellation of singular integrals acting on low regular quadratic and cubic terms allows us to bootstrap to achieve the control of the evolution of the curvature ($\nabla^2 u \in L^1(0,T;L^\infty)$).

Finally, we continue the bootstrapping process to show a new proof for the propagation of regularity with initial $C^{2+\gamma}$ patches. Describing the dynamics of the patch by a level set, one can get advantage of the extra regularity in the tangential direction. In particular, in checking the evolution of this extra regularity one just needs to control the tangential direction of $\nabla^2 u$. Exploiting the smoothing properties of the Newtonian potential and the persistence of the regularity for the curvature, we are able to prove the propagation of $C^{2+\gamma}$ regularity. We realize that it is possible to find that extra cancellation dealing directly with singular integral operators.

The structure of the chapter is as follows: In next section we show that the weak formulation we use to understand the solutions satisfies indeed the expected physical conditions at the interface. In Section 3.3 we prove the persistence of regularity for $C^{1+\gamma}$ patches and $u_0 \in H^{\gamma+s}$. In Section 3.4 we show further that the curvature of the patches remains bounded. Finally, in Section 3.5 a proof of the propagation of $C^{2+\gamma}$ regularity in which we deal directly with the explicit expression of the tangential second derivatives of u is given.

3.2 Weak solutions and physical conditions

In this section we first state the definition of weak solutions for the system (3.1)-(3.2). Later we show that under suitable regularity assumptions these solutions satisfy the naturally expected physical conditions (see e.g. [51] and the references therein):

- The interface moves with the fluid (no mass transfer):

$$\begin{aligned} z_t(\alpha,t)\cdot n(\alpha,t) &= u\left(z(\alpha,t),t\right)\cdot n(\alpha,t), \\ z(\alpha,0) &= z_0(\alpha), \end{aligned} \tag{3.3}$$

where z_0 is a parametrization of the boundary of the patch ∂D_0.

- Continuity of the velocity at the interface:

$$[u]|_{\partial D(t)} := \lim_{\substack{x\to x_0\in\partial D(t),\\ x\in D(t)}} u(x) - \lim_{\substack{x\to x_0\in\partial D(t),\\ x\in D(t)^c}} u(x) = 0, \tag{3.4}$$

- Continuity of the stress tensor at the interface:

$$[\mathcal{T}\cdot n]\,|_{\partial D(t)} = 0, \tag{3.5}$$

where $\mathcal{T} = -p\,\mathbb{I} + (\nabla u + \nabla u^*)$ and ∇u^* denotes transpose of ∇u.

Definition 3.2.1. *We say that (ρ,u,p) is a weak solution of the system (3.1)-(3.2) provided that $\forall\varphi\in C_c^\infty([0,T);C_c^\infty(\mathbb{R}^2))$,*

$$\int_0^T\int_{\mathbb{R}^2}\nabla\varphi\cdot u\,dxdt = 0, \tag{3.6}$$

$$\int_{\mathbb{R}^2}\varphi(0)\rho_0 dx + \int_0^T\int_{\mathbb{R}^2}\rho D_t\varphi dx, \tag{3.7}$$

and that for all $\phi\in(C_c^\infty([0,T);C_c^\infty(\mathbb{R}^2)))^2$

$$\int_{\mathbb{R}^2}\phi(0)\cdot\rho_0u_0\,dx+\int_0^T\int_{\mathbb{R}^2}D_t\phi\cdot\rho u\,dxdt-\int_0^T\int_{\mathbb{R}^2}\nabla\phi:(\nabla u+\nabla u^*)\,dxdt+\int_0^T\int_{\mathbb{R}^2}p\nabla\cdot\phi\,dxdt = 0, \tag{3.8}$$

where $\nabla\phi:\nabla u = \sum_{i,j=1}^2\partial_i\varphi_j\partial_i u_j$.

Proposition 3.2.2. *Let (ρ,u,p) be a weak solution of (3.1)-(3.2) with initial data as in Theorem 3.3.1. Then, the conditions (3.3)-(3.5) hold.*

Proof: The weak incompressibility condition (3.6) implies the continuity of the normal velocity at the interface, which jointly to the mass conservation (3.7) yields the interface dynamics condition (3.3) (see e.g. [38]). Moreover, the results given in Theorem 3.3.1 for initial data in $H^{\gamma+s}$ gives that $u\in L^1(0,T;C^{1+\gamma})$, hence the velocity is continuous also in the tangential direction for a.e. $t>0$.

We show then the continuity of the stress tensor. We can write

$$\begin{aligned} 0 = &\int_{\mathbb{R}^2}\phi(0)\rho_0u_0\,dx + \int_0^T\int_{D(t)\cup D(t)^c}D_t\phi\cdot\rho u\,dxdt \\ &-\int_0^T\int_{D(t)\cup D(t)^c}\nabla\phi\cdot(\nabla u+\nabla u^*)\,dxdt + \int_0^T\int_{D(t)\cup D(t)^c}p\nabla\cdot\phi dxdt. \end{aligned} \tag{3.9}$$

Taking into account that the normal velocity is continuous at the interface and (3.3), the regularity provided by Theorem 3.3.1 allows us to integrate by parts to find that

$$
\begin{aligned}
0 =& \int_0^T\int_{D(t)} \phi\cdot(-\rho_1 D_t u+\Delta u-\nabla p)\,dxdt + \int_0^T\int_{D(t)^c} \phi\cdot(-\rho_2 D_t u+\Delta u-\nabla p)\,dxdt \\
&- \int_0^T\int_{\partial D(t)} \phi n\cdot(\nabla u_1+\nabla u_1^*)\,d\sigma + \int_0^T\int_{\partial D(t)} \phi n\cdot(\nabla u_2+\nabla u_2^*)\,d\sigma \\
&+ \int_0^T\int_{\partial D(t)} \phi p_1 n\,d\sigma - \int_0^T\int_{\partial D(t)} \phi p_2 n\,d\sigma.
\end{aligned}
\tag{3.10}
$$

Thus we deduce that

$$
\int_0^T\int_{\partial D(t)} \phi\left[(p_1-p_2)\,\mathbb{I} - ((\nabla u_1+\nabla u_1^*)-(\nabla u_2+\nabla u_2^*))\right]\cdot n d\sigma = 0. \tag{3.11}
$$

and hence, as p, u are regular enough (3.21), we conclude

$$
[(-p\,\mathbb{I} + (\nabla u+\nabla u^*))\cdot n]\,|_{\partial D(t)} = 0, \qquad \text{a.e. } t\in(0,T). \tag{3.12}
$$

□

3.3 Persistence of $C^{1+\gamma}$ regularity

We present below the theorem that establishes the propagation of regularity for $C^{1+\gamma}$ patches in the case of positive density.

Theorem 3.3.1. *Assume $\gamma\in(0,1)$, $s\in(0,1-\gamma)$, $\rho_1,\rho_2>0$. Let $D_0\subset\mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0\in C^{1+\gamma}$, $u_0\in H^{\gamma+s}$ a divergence-free vector field,*

$$\rho_0(x) = \rho_1 1_{D_0}(x) + \rho_2 1_{D_0^c}(x),$$

and 1_{D_0} the characteristic function of D_0. Then, there exists a unique global solution (u,ρ) of (3.1)-(3.2) such that

$$\rho(x,t) = \rho_1 1_{D(t)}(x) + \rho_2 1_{D(t)^c}(x) \ \text{ and } \ \partial D\in C([0,T];C^{1+\gamma}),$$

where $D(t)=X(D_0,t)$ with X the particle trajectories associated to the velocity field. Moreover,

$$
\begin{gathered}
u\in C([0,T];H^{\gamma+s})\cap L^1(0,T;C^{1+\gamma+\tilde{s}}),\\
t^{\frac{1-(\gamma+s)}{2}}u\in L^\infty(0,T;H^1),\ t^{\frac{2-(\gamma+s)}{2}}u\in L^\infty(0,T;H^2),\\
t^{\frac{2-(\gamma+s)}{2}}u_t\in L^\infty([0,T];L^2)\cap L^2([0,T];H^1),
\end{gathered}
$$

for any $T>0$, $\tilde{s}\in(0,s)$.

Proof: First, as $u_0 \in H^{\gamma+s}$ and $0 < \min\{\rho_1, \rho_2\} < \rho_0 < \max\{\rho_1, \rho_2\} < \infty$, the results in [100] yield the following estimates for any $T \geq 0$:

$$\begin{aligned} A_0(T) &\leq C(\|u_0\|_{L^2}), \\ A_1(T) &\leq C(\|u_0\|_{H^{\gamma+s}}), \\ A_2(T) &\leq C(\|u_0\|_{H^{\gamma+s}}), \\ \int_0^T \|\nabla u\|_{L^\infty} dt &\leq C(T, \|u_0\|_{H^{\gamma+s}}), \end{aligned} \tag{3.13}$$

where the constant C also depends on ρ_1, ρ_2, and A_0, A_1, A_2 are defined by

$$\begin{aligned} A_0(T) &= \sup_{[0,T]} \|\sqrt{\rho} u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt, \\ A_1(T) &= \sup_{[0,T]} t^{1-(\gamma+s)} \|\nabla u\|_{L^2}^2, \\ A_2(T) &= \sup_{[0,T]} t^{2-(\gamma+s)} \left(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 \right) + \int_0^T t^{2-(\gamma+s)} \|\nabla u_t\|_{L^2}^2. \end{aligned}$$

In particular, we note that by interpolation we get

$$\|u\|_{H^{1+\gamma+\tilde{s}}} \leq \|u\|_{H^1}^{1-\gamma-\tilde{s}} \|u\|_{H^2}^{\gamma+\tilde{s}} \leq c t^{-\frac{1-\gamma-s}{2}(1-\gamma-\tilde{s}) - \frac{2-\gamma-s}{2}(\gamma+\tilde{s})},$$

and therefore

$$u \in L^p(0, T; H^{1+\gamma+\tilde{s}}), \qquad p \in [1, 2/(1-(s-\tilde{s}))). \tag{3.14}$$

Proceeding by interpolation again, it follows that

$$\int_0^T \|u_t\|_{H^{\gamma+\tilde{s}}}^q dt \leq \int_0^T \left(\|u_t\|_{L^2}^{1-\gamma-\tilde{s}} \|u_t\|_{H^1}^{\gamma+\tilde{s}} \right)^q dt \leq c \int_0^T t^{-\frac{2-\gamma-s}{2}(1-\gamma-\tilde{s})q} \frac{\|u_t\|_{H^1}^{q(\gamma+\tilde{s})} t^{\frac{2-\gamma-s}{2}(\gamma+\tilde{s})q}}{t^{\frac{2-\gamma-s}{2}(\gamma+\tilde{s})q}} dt,$$

hence by Hölder inequality we conclude

$$u_t \in L^q(0, T; H^{\gamma+\tilde{s}}), \qquad q \in [1, 2/(2-(s-\tilde{s}))). \tag{3.15}$$

Next, we rewrite (3.2) as a forced heat equation

$$u_t - \Delta u = -\rho u \cdot \nabla u + (1-\rho) u_t - \nabla p.$$

We apply first the Leray projector $\mathbb{P} = \mathrm{I}_2 - \nabla \Delta^{-1}(\nabla \cdot\ )$ to obtain

$$u_t - \Delta u = -\mathbb{P}(\rho u \cdot \nabla u) + \mathbb{P}((1-\rho) u_t), \tag{3.16}$$

and denote

$$\begin{gathered} u = v_1 + v_2 + v_3, \\ v_1 = e^{t\Delta} u_0, \qquad v_2 = -(\partial_t - \Delta)_0^{-1} \mathbb{P}(\rho u \cdot \nabla u), \qquad v_3 = (\partial_t - \Delta)_0^{-1} \mathbb{P}((1-\rho) u_t). \end{gathered} \tag{3.17}$$

Recalling the following particular case of Gagliardo-Nirenberg inequality

$$\|f\|_{L^r(\mathbb{R}^2)} \leq c\|f\|_{L^2}^{2/r}\|\nabla f\|_{L^2}^{1-2/r}, \qquad r \in [2,\infty), \tag{3.18}$$

we deduce from (3.13) that

$$\|u\cdot\nabla u\|_{L^2} \leq \|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^{1/2} \leq ct^{-1+\frac{3}{4}(\gamma+s)}.$$

Therefore, it follows that

$$\|v_2\|_{L^2} + \|v_3\|_{L^2} \leq c\left(t^{-1+\frac{3}{4}(\gamma+s)} + t^{-1+\frac{1}{2}(\gamma+s)}\right).$$

Then, the splitting in (3.17) provides

$$\|u\|_{L^\infty_T(H^{\gamma+s})} \leq c\|u_0\|_{H^{\gamma+s}} + \|(1-\Delta)^{\frac{\gamma+s}{2}}(v_2+v_3)\|_{L^\infty(L^2)},$$

so that using the decay properties of the heat kernel (see e.g. Appendix D in [102]) and Young's inequality for convolutions we obtain

$$\begin{aligned}\|u\|_{L^\infty_T(H^{\gamma+s})} &\leq c\|u_0\|_{H^{\gamma+s}}+c\left(\|u_0\|_{H^{\gamma+s}}\right)\left\|\int_0^t \|(1-\Delta)^{\frac{\gamma+s}{2}}K(t-\tau)\|_{L^1}(\|v_2\|_{L^2}+\|v_3\|_{L^2})d\tau\right\|_{L^\infty_T}\\ &\leq c\|u_0\|_{H^{\gamma+s}}+c\left(\|u_0\|_{H^{\gamma+s}}\right)\left\|\int_0^t \frac{\tau^{-1+\frac{3}{4}(\gamma+s)}+\tau^{-1+\frac{1}{2}(\gamma+s)}}{(t-\tau)^{\frac{\gamma+s}{2}}}d\tau\right\|_{L^\infty_T} \leq c\left(\|u_0\|_{H^{\gamma+s}}\right).\end{aligned} \tag{3.19}$$

Notice that from (3.13) the velocity field satisfies $u \in L^1(0,T;W^{1,\infty})$, so the initial density is transported and remains as a patch

$$\rho(x,t) = \rho_1 1_{D(t)}(x) + \rho_2 1_{D(t)^c}(x)$$

with Lipschitz boundary. We write (3.2) as follows

$$\Delta u = \mathbb{P}\left(\rho D_t u\right). \tag{3.20}$$

From (3.14) we deduce $\nabla\cdot(u\otimes u) \in L^{p/2}(0,T;H^{\gamma+\tilde{s}})$. This joined to (3.15) yields that $D_t u \in L^q(0,T;H^{\gamma+\tilde{s}})$ for $q \in [1, 2/(2-(s-\tilde{s})))$. By Sobolev embeddings we have that

$$D_t u \in L^q(0,T;L^{\frac{2}{1-(\gamma+\tilde{s})}}),$$

and therefore

$$\rho D_t u \in L^q(0,T;L^{\frac{2}{1-(\gamma+\tilde{s})}}).$$

Finally, recalling that the Leray projector is bounded in L^p spaces, $1<p<\infty$, we take the inverse of the Laplacian in (3.20) to find that

$$\|u\|_{\dot{W}^{2,\frac{2}{1-(\gamma+\tilde{s})}}} \leq c\|\Delta^{-1}(\rho D_t u)\|_{\dot{W}^{2,\frac{2}{1-(\gamma+\tilde{s})}}} \leq c\|\rho D_t u\|_{L^{\frac{2}{1-(\gamma+\tilde{s})}}}.$$

From (3.14) it is clear that $u \in L^q(0,T;L^\infty)$, therefore we conclude by Sobolev embedding in Hölder spaces that $u \in L^q(0,T;C^{1+\gamma+\tilde{s}})$. Hence, the regularity of the patch is propagated as explained in the lines following (2.13), yielding the persistence of $C^{1+\gamma}$ regularity $\|z\|_{L^\infty(0,T;C^{1+\gamma})} \leq C(T)$.

□

Remark 3.3.2. *From the momentum equation it is easy to see that*

$$\|\Delta u\|_{H^\mu}^2 + \|\nabla p\|_{H^\mu}^2 = \|\rho D_t u\|_{H^\mu}^2.$$

Noticing that $\rho \in \mathcal{M}(H^\sigma)$, $\sigma \in (-1/2, 1/2)$ *and recalling that* $D_t u \in L^q(0,T; H^{\gamma+s})$, *it follows that*

$$p \in L^q(0,T; \dot{H}^{1+\mu}), \quad u \in L^q(0,T; H^{2+\mu}), \quad \mu < \min\{1/2, \gamma+s\}. \tag{3.21}$$

3.4 Persistence of $W^{2,\infty}$ regularity

In this section we show that the curvature of the patch is bounded for all time if initially has $W^{2,\infty}$ boundary.

Theorem 3.4.1. *Assume* $s \in (0,1)$, $\rho_1, \rho_2 > 0$. *Let* $D_0 \subset \mathbb{R}^2$ *be a bounded simply connected domain with boundary* $\partial D_0 \in W^{2,\infty}$, $u_0 \in H^{1+s}$ *a divergence-free vector field,*

$$\rho_0(x) = \rho_1 1_{D_0}(x) + \rho_2 1_{D_0^c}(x),$$

and 1_{D_0} *the characteristic function of* D_0. *Then, there exists a unique global solution* (u,ρ) *of* (3.1)-(3.2) *such that*

$$\rho(x,t) = \rho_1 1_{D(t)}(x) + \rho_2 1_{D(t)^c}(x) \text{ and } \partial D \in C([0,T]; W^{2,\infty}),$$

where $D(t) = X(D_0, t)$ *with* X *the particle trajectories associated to the velocity field. Moreover,*

$$\begin{aligned} u &\in C([0,T]; H^{1+s}) \cap L^2(0,T; H^{2+\mu}) \cap L^p(0,T; W^{2,\infty}),\\ t^{\frac{1-s}{2}} u_t &\in L^\infty([0,T]\,; L^2) \cap L^2([0,T]\,; H^1),\\ t^{\frac{2-s}{2}} D_t u &\in L^\infty([0,T]\,; H^1) \cap L^2([0,T]\,; H^2), \end{aligned}$$

for any $T > 0$, $\mu < \min\{1/2, s\}$ *and* $p \in [1, 2/(2-s))$. *If* $s < 1/2$ *it also holds that* $u \in L^q(0,T; H^{2+\delta})$ *for any* $\delta \in (s, 1/2)$ *with* $q \in [1, 2/(1+\delta-s))$.

Proof: First, we notice that once we get $u \in L^p(0,T; W^{2,\infty})$ the propagation of regularity for the patch follows by considering the particle trajectories associated to the flow. Hence, we proceed to prove that the velocity belongs to that space.

3.4.1 Regularity of u_t

We start by proving in this section that $t^{\frac{1-s}{2}} u_t \in L^\infty([0,T]\,; L^2) \cap L^2([0,T]\,; H^1)$. As before, it is easy to get

$$\|\sqrt{\rho} u\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 d\tau \leq c\|u_0\|_{L^2}^2. \tag{3.22}$$

We now take inner product with u_t and use Young's inequality to obtain

$$\|\sqrt{\rho} u_t\|_{L^2}^2 + \frac{d}{dt}\|\nabla u\|_{L^2}^2 \leq c\|u\|_{L^4}^2 \|\nabla u\|_{L^4}^2. \tag{3.23}$$

Using (3.18) with $r = 4$, from the velocity equation and Young's inequality one infers that

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 \leq c\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|u\|_{L^2}^2\|\nabla u\|_{L^2}^4\right). \tag{3.24}$$

Hence, applying (3.18) again in (3.23) and using (3.24), we get

$$\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{d}{dt}\|\nabla u\|_{L^2}^2 \leq c\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4,$$

so by Grönwall's inequality and (3.22) we conclude that

$$\|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2 d\tau \leq \|\nabla u_0\|_{L^2}^2 e^{c\|u_0\|_{L^2}t}.$$

We can close the estimates for u and u_t at this level of regularity:

$$\begin{gathered}\|u\|_{L^\infty_T(L^2)} + \|u\|_{L^2_T(H^1)} \leq c\|u_0\|_{L^2},\\ \|u\|_{L^\infty_T(H^1)} + \|u\|_{L^2_T(H^2)} + \|u_t\|_{L^2_T(L^2)} \leq c\left(\|u_0\|_{L^2}\right)\|u_0\|_{H^1}.\end{gathered} \tag{3.25}$$

From this last estimate, (3.24) rewrites as

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2 \leq c\left(\|\sqrt{\rho}u_t\|_{L^2}^2 + 1\right). \tag{3.26}$$

We proceed next by an interpolation argument. First, we consider the linear momentum equation for v

$$\rho v_t + \rho u\cdot\nabla v - \Delta v + \nabla p = 0, \qquad \rho_t + u\cdot\nabla\rho = 0. \tag{3.27}$$

By previous arguments it follows that

$$\|v\|_{L^\infty_T(H^1)}^2 + \|\sqrt{\rho}v_t\|_{L^2_T(L^2)}^2 \leq c\left(\|u_0\|_{L^2}\right)\|v_0\|_{H^1}^2, \tag{3.28}$$

$$\|\nabla^2 v\|_{L^2}^2 \leq c\left(\|\sqrt{\rho}v_t\|_{L^2}^2 + \|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla v\|_{L^2}^2\right),$$

and hence

$$\|\nabla^2 v\|_{L^2}^2 \leq c\left(\|\sqrt{\rho}v_t\|_{L^2}^2 + \|v_0\|_{H^1}^2\right), \tag{3.29}$$

Derivation in time of (3.27) yields the following equation

$$\rho v_{tt} + \rho u\cdot\nabla v_t - \Delta v_t + \nabla p_t = -\rho_t v_t - \rho_t u\cdot\nabla v - \rho u_t\cdot\nabla v,$$

and thus we obtain

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}v_t\|_{L^2}^2 + \|\nabla v_t\|_{L^2}^2 = -\int_{\mathbb{R}^2}\rho_t|v_t|^2dx - \int_{\mathbb{R}^2}\rho_t u\cdot\nabla v\cdot v_t dx - \int_{\mathbb{R}^2}\rho u_t\cdot\nabla v\cdot v_t dx,$$

where we have used that $\rho_t = -u\cdot\nabla\rho$. Multiplication by the weight t and integration in time implies that

$$\frac{t}{2}\|\sqrt{\rho}v_t\|_{L^2}^2 + \int_0^t \tau\|\nabla v_t\|_{L^2}^2 d\tau = I_1 + I_2 + I_3 + I_4, \tag{3.30}$$

where

$$I_1 = \frac{1}{2}\int_0^t \|\sqrt{\rho}v_t\|_{L^2}^2 d\tau, \qquad I_2 = -\int_0^t \tau \int_{\mathbb{R}^2} \rho_t |v_t|^2 dx d\tau,$$
$$I_3 = -\int_0^t \tau \int_{\mathbb{R}^2} \rho_t u \cdot \nabla v \cdot v_t dx d\tau, \qquad I_4 = -\int_0^t \tau \int_{\mathbb{R}^2} \rho u_t \cdot \nabla v \cdot v_t dx d\tau.$$

The first term is controlled by (3.28) as follows

$$I_1 \leq c\left(\|u_0\|_{L^2}\right)\|v_0\|_{H^1}^2. \tag{3.31}$$

Recalling that u is divergence-free and that $\rho_t = -u \cdot \nabla\rho$, integration by parts in I_2 yields the following

$$I_2 \leq \int_0^t \tau \int_{\mathbb{R}^2} \rho u \cdot \nabla |v_t|^2 dx d\tau \leq c \int_0^t \tau \|\nabla v_t\|_{L^2}\|u\|_{L^4}\|v_t\|_{L^4} d\tau.$$

By (3.18) and Young's inequality I_2 is bounded by

$$I_2 \leq \frac{1}{10}\int_0^t \tau \|\nabla v_t\|_{L^2}^2 d\tau + c(\|u_0\|_{L^2})\int_0^t \tau \|\sqrt{\rho}v_t\|_{L^2}^2 \|\nabla u\|_{L^2}^2 d\tau.$$

Using again (3.18) and (3.29) we get that

$$\begin{aligned} I_4 \leq\; & c\left(\|u_0\|_{H^1}\right)\int_0^t \tau \|\sqrt{\rho}u_t\|_{L^2}\|\nabla v\|_{L^2}^{1/2}\|\sqrt{\rho}v_t\|_{L^2}\|\nabla v_t\|_{L^2}^{1/2} d\tau \\ & + c\left(\|u_0\|_{H^1}\right)\int_0^t \tau \|\sqrt{\rho}u_t\|_{L^2}\|\nabla v\|_{L^2}\|\sqrt{\rho}v_t\|_{L^2}^{1/2}\|\nabla v_t\|_{L^2}^{1/2} d\tau, \end{aligned}$$

and therefore by (3.28) and Young's inequality it follows that

$$\begin{aligned} I_4 \leq\; & \frac{1}{10}\int_0^t \tau \|\nabla v_t\|_{L^2}^2 d\tau + c\left(\|u_0\|_{H^1}\right)\int_0^t \tau \|\sqrt{\rho}u_t\|_{L^2}^2 \|\sqrt{\rho}v_t\|_{L^2}^2 d\tau \\ & + c\left(\|u_0\|_{L^2}, T\right)\|v_0\|_{H^1}^2. \end{aligned} \tag{3.32}$$

After integration by parts, I_3 is decomposed as follows

$$I_3 = I_{31} + I_{32} + I_{33},$$

where

$$I_{31} = -\int_0^t \tau \int_{\mathbb{R}^2} \rho v_t \cdot \nabla u \cdot \nabla v \cdot u \, dx d\tau,$$
$$I_{32} = -\int_0^t \tau \int_{\mathbb{R}^2} \rho(u \otimes u) : \nabla^2 v \cdot v_t d\tau dx,$$
$$I_{33} = -\int_0^t \tau \int_{\mathbb{R}^2} \rho(u \cdot \nabla v) \cdot (u \cdot \nabla v_t) dx d\tau.$$

First we use again (3.18) and Young's inequality to obtain

$$\begin{aligned} I_{31} &\le \int_0^t \tau \|\rho u\cdot\nabla u\|_{L^2}\|v_t\|_{L^4}\|\nabla v\|_{L^4}d\tau \\ &\le \frac{1}{10}\int_0^t \tau\|\nabla v_t\|_{L^2}^2 d\tau + c\int_0^t \tau\|u\cdot\nabla u\|_{L^2}^{4/3}\|\sqrt{\rho}v_t\|_{L^2}^{2/3}\|\nabla v\|_{L^2}^{2/3}\|\nabla^2 v\|_{L^2}^{2/3}d\tau. \end{aligned}$$

After using (3.29), Young's inequality and the previous estimates on u and v yield

$$\begin{aligned} I_{31} \le \frac{1}{10}\int_0^t \tau\|\nabla v_t\|_{L^2}^2 d\tau + c\left(\|u_0\|_{H^1}\right)\int_0^t \tau\|\sqrt{\rho}v_t\|_{L^2}^2\left(1+\|\nabla^2 u\|_{L^2}^2\right)d\tau \\ + c\left(\|u_0\|_{L^2},T\right)\|v_0\|_{H^1}^2. \end{aligned}$$

Since (3.18) and (3.25) give $\|u\|_{L^8} \le c\left(\|u_0\|_{L^2}\right)\|u_0\|_{H^1}$, using (3.18), (3.29) and Young's inequality the terms I_{32} and I_{33} are bounded as I_{31}. Therefore,

$$\begin{aligned} I_3 \le \frac{1}{10}\int_0^t \tau\|\nabla v_t\|_{L^2}^2 d\tau + c\int_0^t \tau\|\sqrt{\rho}v_t\|_{L^2}^2\left(1+\|\nabla^2 u\|_{L^2}^2\right)d\tau \\ + c\left(\|u_0\|_{L^2},T\right)\|v_0\|_{H^1}^2. \end{aligned} \tag{3.33}$$

Joining the above bounds (3.31)-(3.33) we get from (3.30) that

$$\begin{aligned} t\|\sqrt{\rho}v_t\|_{L^2}^2 + \int_0^t \tau\|\nabla v_t\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{L^2},T\right)\|v_0\|_{H^1}^2 \\ + c\int_0^t \tau\|\sqrt{\rho}v_t\|_{L^2}^2\left(\|\nabla^2 u\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + 1\right)d\tau, \end{aligned}$$

thus by Grönwall's inequality we finally find

$$t\|\sqrt{\rho}v_t\|_{L^2}^2 + \int_0^t \tau\|\nabla v_t\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{H^1},T\right)\|v_0\|_{H^1}^2.$$

We notice that from (3.27) we have $\|\sqrt{\rho}v_t\|_{L^2}(\tau) \le c\left(\|u_0\|_{H^1}\right)\|v\|_{H^2}(\tau)$ for all $\tau \ge 0$, so if we assumed $v_0 \in H^2$, repeating the steps above without weights would lead to

$$\|\sqrt{\rho}v_t\|_{L^2}^2 + \int_0^t \|\nabla v_t\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{H^1},T\right)\|v_0\|_{H^2}^2.$$

Finally, by linear interpolation between the last two inequalities [89] we conclude that

$$t^{1-s}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^t \tau^{1-s}\|\nabla u_t\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{H^1},T\right)\|u_0\|_{H^{1+s}}^2.$$

Using (3.26) we are able to finally find

$$t^{1-s}\|\sqrt{\rho}u_t\|_{L^2}^2 + t^{1-s}\|\nabla^2 u\|_{L^2}^2 + \int_0^t \tau^{1-s}\|\nabla u_t\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{H^1},T\right)\|u_0\|_{H^{1+s}}^2. \tag{3.34}$$

We note that by Sobolev interpolation we have in particular that for $\tilde{s} \in (0,s)$

$$u_t \in L^2(0,T;H^{\tilde{s}}). \tag{3.35}$$

3.4.2 Higher regularity for u

This section is devoted to prove that $u \in L^\infty(0,T;H^{1+s}) \cap L^2(0,T;H^{2+\mu})$. First, we notice that by Theorem 3.3.1 $\rho(t)$ remains as a patch with Lipschitz boundary for all $t \geq 0$ and hence it is known that

$$\rho \in L^\infty(0,T;\mathcal{M}(H^\sigma)) \quad \sigma \in \left(-\frac{1}{2},\frac{1}{2}\right).$$

From this and the estimates (3.25), (3.35) we infer that

$$(1-\rho)u_t, \;\; \rho u\cdot\nabla u \in L^2(0,T;H^\mu), \quad \mu < \min\left\{\frac{1}{2}, s\right\}.$$

Thus, by classical properties of the heat equation applied to (3.16) we conclude that $u \in L^2(0,T;H^{2+\mu})$ with

$$\|u\|_{L^2_T(H^{2+\mu})} \leq c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.36}$$

To get $u \in L^\infty(0,T;H^{1+s})$ we will use that (3.34) and (3.26) implies

$$t^{1-s}\|\sqrt{\rho}u_t\|^2_{L^2} + t^{1-s}\|\sqrt{\rho}u\cdot\nabla u\|^2_{L^2} \leq c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.37}$$

Then, we write the solution of (3.16) as

$$u = e^{t\Delta}u_0 + (\partial_t - \Delta)^{-1}_0\left(\mathbb{P}\left((1-\rho)u_t\right) - \mathbb{P}\left(\rho u\cdot\nabla u\right)\right),$$

thus we have that

$$\|u\|_{L^\infty_T(H^{1+s})} \leq c\|u_0\|_{H^{1+s}} + c\left|\left|(1-\Delta)^{\frac{1+s}{2}}(\partial_t - \Delta)^{-1}_0\left(\mathbb{P}\left((1-\rho)u_t\right) - \mathbb{P}\left(\rho u\cdot\nabla u\right)\right)\right|\right|_{L^\infty_T(L^2)}.$$

Applying Young's inequality for convolution, (3.37) and the decay properties of the heat kernel we get

$$\begin{aligned}\|u\|_{L^\infty_T(H^{1+s})} &\leq c\|u_0\|_{H^{1+s}} + c\left|\left|\int_0^t \|(1-\Delta)^{\frac{1+s}{2}}K(t-\tau)\|_{L^1}\tau^{-\frac{1-s}{2}}d\tau\right|\right|_{L^\infty_T} \\ &\leq c\|u_0\|_{H^{1+s}} + c\left|\left|\int_0^t (t-\tau)^{-\frac{1+s}{2}}\tau^{-\frac{1-s}{2}}d\tau\right|\right|_{L^\infty_T},\end{aligned}$$

so that we conclude

$$\|u\|_{L^\infty_T(H^{1+s})} \leq c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.38}$$

3.4.3 Higher regularity for $D_t u$

We will show that $t^{\frac{2-s}{2}} D_t u \in L^\infty([0,T]\,;H^1) \cap L^2([0,T]\,;H^2)$. Applying D_t to (3.2) yields

$$\rho D_t^2 u - \Delta D_t u + \nabla D_t p = -2\nabla u_i \cdot \partial_i \nabla u + \Delta u \cdot \nabla u - \nabla u^T \cdot \nabla p, \tag{3.39}$$

where Einstein summation convention is used. By definition of $D_t u$, it follows directly from previous estimates (3.34) and (3.38) that

$$t^{1-s}\|\sqrt{\rho} D_t u\|_{L^2}^2 + \int_0^t \tau^{1-s}\|\nabla D_t u\|_{L^2}^2 \le c\left(\|u_0\|_{H^1}, T\right)\|u_0\|_{H^{1+s}}^2. \tag{3.40}$$

In what follows we will denote

$$F = F(\nabla u, \nabla^2 u, \nabla p) = -2\nabla u_i \cdot \partial_i \nabla u + \Delta u \cdot \nabla u - \nabla u^T \cdot \nabla p.$$

As $\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2} \le \|\rho D_t u\|_{L^2}$, using (3.40) we notice that

$$\|F\|_{L^2}^2 \le c\|\nabla u\|_{L^\infty}^2 \|\rho D_t u\|_{L^2}^2 \le c\, t^{-1+s}\|u\|_{H^{2+\epsilon}}^2,$$

so from (3.36) we find

$$\int_0^t \tau^{1-s}\|F\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.41}$$

By taking dot product of (3.39) with $D_t^2 u$ and integrating in time we find

$$\frac{1}{2} t^{2-s}\|\nabla D_t u\|_{L^2}^2 + \int_0^t \tau^{2-s}\|\sqrt{\rho} D_t^2 u\|_{L^2}^2 d\tau = L_1 + L_2 + L_3 + L_4, \tag{3.42}$$

where

$$L_1 = \frac{2-s}{2}\int_0^t \tau^{1-s}\|\nabla D_t u\|_{L^2}^2 d\tau, \qquad L_2 = -\int_0^t \tau^{2-s}\int_{\mathbb{R}^2} D_t^2 u \cdot \nabla D_t p \;\; dx d\tau,$$

$$L_3 = \int_0^t \tau^{2-s}\int_{\mathbb{R}^2} D_t^2 u \cdot F\, dx d\tau, \qquad L_4 = \int_0^t \tau^{2-s}\int_{\mathbb{R}^2} \nabla D_t u : \nabla(u \cdot \nabla D_t u) dx d\tau.$$

The first term, L_1, is bounded by (3.40),

$$L_1 \le c\left(\|u_0\|_{H^{1+s}}^2, T\right). \tag{3.43}$$

while using (3.41) it follows that

$$L_3 \le \frac{1}{6}\int_0^t \tau^{2-s}\|\sqrt{\rho} D_t^2 u\|_{L^2}^2 d\tau + c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.44}$$

Noticing that $\nabla \cdot D_t^2 u = \nabla \cdot (u \cdot \nabla u_t + D_t u \cdot \nabla u + u \cdot D_t \nabla u)$, integration by parts twice in L_2 yields the following

$$\begin{aligned} L_2 &= \int_0^t \tau^{2-s}\int_{\mathbb{R}^2} \left(\nabla \cdot D_t^2 u\right) D_t p\, dx d\tau = \int_0^t \tau^{2-s}\int_{\mathbb{R}^2} \nabla \cdot (u \cdot \nabla u_t + D_t u \cdot \nabla u + u \cdot D_t \nabla u) D_t p\, dx d\tau \\ &= -\int_0^t \tau^{2-s}\int_{\mathbb{R}^2} (u \cdot \nabla u_t + D_t u \cdot \nabla u + u \cdot D_t \nabla u) \cdot \nabla D_t p\, dx d\tau \\ &\le \int_0^t \tau^{2-s}\|u \cdot \nabla u_t + D_t u \cdot \nabla u + u \cdot D_t \nabla u\|_{L^2}\|\nabla D_t p\|_{L^2} d\tau. \end{aligned} \tag{3.45}$$

We now use the equation (3.39) to estimate the high-order term $\nabla D_t p$. First, we notice that

$$\mathbb{P}\Delta D_t u = \Delta D_t u - \nabla\nabla\cdot D_t u, \tag{3.46}$$

so that the equation rewrites as

$$-\mathbb{P}\Delta D_t u + \nabla D_t p = \nabla\nabla\cdot D_t u - \rho D_t^2 u + F.$$

Therefore we obtain

$$\|\mathbb{P}\Delta D_t u\|_{L^2} + \|\nabla D_t p\|_{L^2} \le c\|\sqrt{\rho}D_t^2 u\|_{L^2} + \|F\|_{L^2} + \|\nabla\nabla\cdot D_t u\|_{L^2},$$

and using (3.46) we write

$$\|\Delta D_t u\|_{L^2} + \|\nabla D_t p\|_{L^2} \le c\|\sqrt{\rho}D_t^2 u\|_{L^2} + \|F\|_{L^2} + 2\|\nabla\nabla\cdot D_t u\|_{L^2}. \tag{3.47}$$

If we denote

$$G = u\cdot\nabla u_t + D_t u\cdot\nabla u + u\cdot D_t\nabla u,$$

going back to (3.45), estimate (3.47) provides the following

$$L_2 \le c\int_0^t \tau^{2-s}\|\sqrt{\rho}D_t^2 u\|_{L^2}\|G\|_{L^2}d\tau + \int_0^t \tau^{2-s}\left(\|F\|_{L^2} + 2\|\nabla\nabla\cdot D_t u\|_{L^2}\right)\|G\|_{L^2}d\tau.$$

By Young's inequality it is possible to obtain

$$L_2 \le \frac{1}{6}\int_0^t \tau^{2-s}\|\sqrt{\rho}D_t^2 u\|_{L^2}^2 + c\int_0^t \tau^{2-s}\left(\|F\|_{L^2}^2 + \|\nabla\nabla\cdot D_t u\|_{L^2}^2 + \|G\|_{L^2}^2\right)d\tau. \tag{3.48}$$

The incompressibility condition yields

$$\|\nabla\nabla\cdot D_t u\|_{L^2}^2 = \|\nabla(\nabla u\cdot\nabla u^*)\|_{L^2}^2 \le c\|\nabla u\cdot\nabla^2 u\|_{L^2}^2 \le c\|u\|_{H^{2+\epsilon}}^2\|\nabla^2 u\|_{L^2}^2.$$

Hence from (3.34) and (3.36) we find that

$$\int_0^t \tau^{1-s}\|\nabla\nabla\cdot D_t u\|_{L^2}^2 \le c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.49}$$

On the other hand, the bound (3.38) allows us to write

$$t^{1-s}\|G\|_{L^2}^2 \le c\,t^{1-s}\left(\|\nabla u_t\|_{L^2}^2 + \|D_t u\|_{L^2}^2\|u\|_{H^{2+\epsilon}}^2 + \|\nabla^2 u\|_{L^2}^2\right).$$

which joined to (3.40) and (3.34) yields

$$t^{1-s}\|G\|_{L^2}^2 \le c\left(t^{1-s}\|\nabla u_t\|_{L^2}^2 + \|u\|_{H^{2+\epsilon}}^2 + 1\right),$$

so we conclude using again (3.34) and (3.36) that

$$\int_0^t \tau^{1-s}\|G\|_{L^2}^2 d\tau \le c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.50}$$

If we introduce the bounds (3.41), (3.49) and (3.50) in (3.48) we get that

$$L_2 \leq \frac{1}{6}\int_0^t \tau^{2-s}\|\sqrt{\rho}D_t^2 u\|_{L^2}^2 + c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.51}$$

Finally, the term L_4 is bounded by

$$L_4 \leq \int_0^t \tau^{2-s}\|\nabla D_t u\|_{L^2}\|u\|_{H^{1+\epsilon}}\|\nabla^2 D_t u\|_{L^2}d\tau \leq c\int_0^t \tau^{2-s}\|\nabla D_t u\|_{L^2}\|\nabla^2 D_t u\|_{L^2}d\tau,$$

taking into account (3.38). Estimate (3.47) gives

$$L_4 \leq c\int_0^t \tau^{2-s}\|\nabla D_t u\|_{L^2}\|\sqrt{\rho}D_t^2 u\|_{L^2}d\tau + c\int_0^t \tau^{2-s}\|\nabla D_t u\|_{L^2}\left(\|F\|_{L^2} + 2\|\nabla\nabla\cdot D_t u\|_{L^2}\right)d\tau.$$

As in the bound of L_2, by Young's inequality we have that

$$L_4 \leq \frac{1}{6}\int_0^t \tau^{2-s}\|\sqrt{\rho}D_t^2 u\|_{L^2}^2 + c\int_0^t \tau^{2-s}\|\nabla D_t u\|_{L^2}^2 + c\left(\|u_0\|_{H^{1+s}}, T\right),$$

and (3.40) implies

$$L_4 \leq \frac{1}{6}\int_0^t \tau^{2-s}\|\sqrt{\rho}D_t^2 u\|_{L^2}^2 + c\left(\|u_0\|_{H^{1+s}}, T\right). \tag{3.52}$$

Introducing the bounds (3.43), (3.44), (3.51) and (3.52) in (3.42), we conclude that

$$t^{2-s}\|\nabla D_t u\|_{L^2}^2 + \int_0^t \tau^{2-s}\|\sqrt{\rho}D_t^2 u\|_{L^2}^2 d\tau \leq c\left(\|u_0\|_{H^{1+s}}, T\right).$$

Recalling (3.47), (3.41) and (3.49) we find in addition that

$$\int_0^t \tau^{2-s}\|D_t u\|_{H^2}^2 d\tau \leq c\left(\|u_0\|_{H^{1+s}}, T\right).$$

By Sobolev interpolation we note that in particular we have for $\tilde{s} \in (0, s)$,

$$D_t u \in L^p(0, T; H^{1+\tilde{s}}), \qquad 1 \leq p < \frac{2}{2-(s-\tilde{s})}. \tag{3.53}$$

3.4.4 Critical regularity for u

In this section we will conclude that $u \in L^p(0, T; W^{2,\infty})$. From (3.20) we have that

$$\nabla^2 u = \nabla^2\Delta^{-1}\mathbb{P}\left(\rho D_t u\right),$$

where $\mathbb{P}f_i = f_i - R_i R_j f_j$. The operators $\nabla^2\Delta^{-1}\mathbb{P}$ are Fourier multipliers and therefore can be written as convolutions

$$\partial_k\partial_l\Delta^{-1}\mathbb{P}f_i(x) = \left(K_{klij} \star f_j\right)(x), \tag{3.54}$$

with kernels given by

$$K_{klij}(x) = \mathcal{F}^{-1}\left(\frac{\xi_k\xi_l}{|\xi|^2}\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)\right)(x). \tag{3.55}$$

By symmetries it suffices to consider the following three cases:

$$\begin{aligned}
\partial_1^2\Delta^{-1}\mathbb{P}f_1(x) &= \left(\tilde{K}_{111j} \star f_j\right)(x) + \frac{1}{8}f_1(x),\\
\partial_1^2\Delta^{-1}\mathbb{P}f_2(x) &= \left(\tilde{K}_{112j} \star f_j\right)(x) + \frac{3}{8}f_2(x),\\
\partial_1\partial_2\Delta^{-1}\mathbb{P}f_1(x) &= \left(\tilde{K}_{121j} \star f_j\right)(x) - \frac{1}{8}f_2(x).
\end{aligned}$$

where the kernels $\tilde{K}_{klij}$ are even and have zero mean on circles. They can be computed explicitly as they correspond to sums of second and fourth order Riesz transforms (see Chapter 3.3 in [111]). For simplicity we will denote by K the kernels $\tilde{K}_{klij}$ and we rewrite the above equations as singular integral operators plus identities as follows

$$\nabla^2 u = \mathrm{SIO}\left(\rho D_t u\right) + c\rho D_t u. \tag{3.56}$$

Then we decompose as follows

$$\begin{aligned}
\mathrm{SIO}\left(\rho D_t u\right) = {} & \rho_2\,\mathrm{SIO}\left(D_t u\right)\\
& + (\rho_1 - \rho_2)\int_{D(t)} K(x-y)\cdot\left(D_t u(y,t) - D_t u(x,t)\right)dy\\
& + (\rho_1 - \rho_2)\,\mathrm{SIO}\left(1_{D(t)}\right)D_t u(x,t) = M_1 + M_2 + M_3.
\end{aligned}$$

By Sobolev embedding and (3.53) we get that

$$D_t u \in L^p(0,T;C^{\tilde{s}}), \tag{3.57}$$

hence we deduce that

$$|M_1| \le c\left(\|D_t u\|_{C^{\tilde{s}}}(t) + \|D_t u\|_{L^2}(t)\right),$$

and analogously

$$|M_2| \le c\|D_t u\|_{C^{\tilde{s}}}(t).$$

As by Theorem 3.3.1 $\rho(t)$ is a $C^{1+\gamma}$ patch for all $t \ge 0$, $\gamma \in (0,1)$, and the fact that the kernels in the singular integral operators are even, it is possible to obtain (see [11] for more details)

$$|M_3| \le c\|D_t u\|_{L^\infty}(t).$$

We therefore conclude that

$$\|\mathrm{SIO}\left(\rho D_t u\right)\|_{L^p_T(L^\infty)} \le c\left(\|u_0\|_{H^{1+s}}, T\right), \tag{3.58}$$

and hence (3.56) gives

$$u \in L^p(0,T;W^{2,\infty}), \qquad 1 \le p < \frac{2}{2-(s-\tilde{s})}.$$

Remark 3.4.2. *We have obtained $u \in L^2(0,T;H^{2+\mu})$ with $\mu < \min\{s,1/2\}$. In the case $s<1/2$ we can also get $u \in L^q(0,T;H^{2+\delta})$ with $\delta \in (s,1/2)$ and $q \in [1,2/(1+\delta-s)) \subset [1,2)$. This is achieved by rewriting the equation as in (3.20) to take advantage of the smoothing properties of the Laplace equation. If $s<1/2$, from $D_t u \in L^1(0,T;H^{1+\tilde{s}}) \cap L^2(0,T;H^{\tilde{s}})$ one finds by interpolation that $D_t u \in L^q(0,T;H^\delta)$. Since ρ is a multiplier in H^δ for any $\delta \in (s,1/2)$, from standard properties of the Laplace equation it follows that*

$$u \in L^q(0,T;H^{2+\delta}).$$

□

3.5 Persistence of $C^{2+\gamma}$ regularity

This section is devoted to show that $C^{2+\gamma}$ regularity is preserved globally in time.

Theorem 3.5.1. *Assume $\gamma \in (0,1)$, $s \in (0,1-\gamma)$, $\rho_1,\rho_2>0$. Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary $\partial D_0 \in C^{2+\gamma}$, $u_0 \in H^{1+\gamma+s}$ a divergence-free vector field and*

$$\rho_0(x) = \rho_1 1_{D_0}(x) + \rho_2 1_{D_0^c}(x).$$

Then, there exists a unique global solution (u,ρ) of (3.1)-(3.2) such that

$$\rho(x,t) = \rho_1 1_{D(t)}(x) + \rho_2 1_{D(t)^c}(x) \text{ and } \partial D \in C([0,T];C^{2+\gamma}).$$

Moreover,

$$\begin{aligned} u &\in C([0,T];H^{1+\gamma+s}) \cap L^2(0,T;H^{2+\mu}) \cap L^p(0,T;W^{2,\infty}),\\ t^{\frac{1-(\gamma+s)}{2}} u_t &\in L^\infty([0,T];L^2) \cap L^2([0,T];H^1),\\ t^{\frac{2-(\gamma+s)}{2}} D_t u &\in L^\infty([0,T];H^1) \cap L^2([0,T];H^2), \end{aligned}$$

for any $T>0$, $\mu < \min\{1/2,\gamma+s\}$ and $p \in [1,2/(2-(\gamma+s)))$. If $\gamma+s<1/2$ it also holds that $u \in L^q(0,T;H^{2+\delta})$ for any $\delta \in (\gamma+s,1/2)$ with $q \in [1,2/(1+\delta-\gamma-s))$.

Proof: Since $\gamma+s \in (0,1)$, the estimates on u, u_t and $D_t u$ follow as in Theorem 3.4.1. We now describe the patch using a level-set function φ:

$$\partial_t\varphi + u\cdot\nabla\varphi = 0, \qquad \varphi(x,0)=\varphi_0(x),$$

$$D_0 = \{x \in \mathbb{R}^2 : \varphi_0(x) > 0\},$$

so that at time t, $D(t) = X(t,D_0) = \{x \in \mathbb{R}^2 : \varphi(x,t)>0\}$. Then, the vector field given by $W(t) = \nabla^\perp\varphi(t)$ is tangent to the patch and evolves as follows

$$\partial_t W + u\cdot\nabla W = W\cdot\nabla u, \quad W(0) = \nabla^\perp\varphi_0. \tag{3.59}$$

To control $C^{2+\gamma}$ regularity of $\partial D(t)$ we shall ensure that ∇W remains in C^γ. By differentiating (3.59) one obtains

$$\partial_t \nabla W + u \cdot \nabla(\nabla W) = W \cdot \nabla^2 u + \nabla W \cdot \nabla u + \nabla u \cdot \nabla W.$$

Since u is Lipschitz, the following estimate holds for all $t \in [0, T]$:

$$\|\nabla W\|_{C^\gamma}(t) \leq \|\nabla W_0\|_{C^\gamma} e^{c\int_0^t \|\nabla u\|_{L^\infty} d\tau} + e^{c\int_0^t \|\nabla u\|_{L^\infty} d\tau} \int_0^t \left(\|W \cdot \nabla^2 u\|_{C^\gamma} + 2\|\nabla W\|_{L^\infty} \|\nabla u\|_{C^\gamma}\right) d\tau.$$

From this and previous estimates we get that

$$\|\nabla W\|_{C^\gamma}(t) \leq c_1(T) + c_2(T) \int_0^t \|W \cdot \nabla^2 u\|_{C^\gamma}(\tau) d\tau.$$

Therefore the result is obtained once we prove that $W \cdot \nabla^2 u \in L^1(0, T; C^\gamma)$.

Applying Fourier transform in (3.54) gives that

$$\mathcal{F}\left(W_k \partial_k \partial_l u_i\right)(\xi) = \hat{W}_k(\xi) \star \mathcal{F}\left(\partial_k \partial_l u_i\right)(\xi) = \hat{W}_k(\xi) \star \left[\frac{\xi_k \xi_l}{|\xi|^2}\left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\right) \mathcal{F}\left(\rho D_t u_j\right)(\xi)\right].$$

Using the notation (3.55), we introduce the following splitting

$$\begin{aligned}
\mathcal{F}\left(W_k \partial_k \partial_l u_i\right)(\xi) = &\left(\hat{K}_{ijkl} \mathcal{F}\left(\rho D_t u_j\right)\right) \star \hat{W}_k(\xi) - \hat{K}_{ijkl}(\xi) \mathcal{F}\left(W_k \rho D_t u_j\right)(\xi) \\
&+ \hat{K}_{ijkl}(\xi) \mathcal{F}\left(W_k \rho D_t u_j\right)(\xi) - \left(\hat{K}_{ijkl} \mathcal{F}\left(\rho W_k\right)\right) \star \mathcal{F}\left(D_t u_j\right)(\xi) \\
&+ \left(\hat{K}_{ijkl} \mathcal{F}\left(\rho W_k\right)\right) \star \mathcal{F}\left(D_t u_j\right)(\xi).
\end{aligned}$$

We note that since W is tangent to the density patch, the last term vanish

$$\hat{K}_{ijkl} \mathcal{F}\left(\rho W_k\right)(\xi) = -i\left(\frac{\xi_l}{|\xi|^2}\left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\right)\right) \mathcal{F}\left(\partial_k\left(W_k \rho\right)\right)(\xi) = 0.$$

Hence the previous splitting writes as

$$W(x, t) \cdot \nabla^2 u(x, t) = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} K(x - y) \cdot \left(W(x, t) - W(y, t)\right) \rho(y, t) D_t u(y, t) dy, \\
I_2 &= \int_{\mathbb{R}^2} K(x - y) \cdot W(y, t) \rho(y, t) \left(D_t u(y, t) - D_t u(x, t)\right) dy.
\end{aligned}$$

The Lemma in Appendix of [11] yields the following

$$\begin{aligned}
\|I_1\|_{C^\gamma} &\leq c\|W\|_{C^\gamma}(t) \left(\|\rho D_t u\|_{L^\infty}(t) + \|\mathrm{SIO}\left(\rho D_t u\right)\|_{L^\infty}(t)\right), \\
\|I_2\|_{C^\gamma} &\leq c\|D_t u\|_{C^\gamma}(t) \left(\|\rho W\|_{L^\infty}(t) + \|\mathrm{SIO}\left(\rho W\right)\|_{L^\infty}(t)\right).
\end{aligned}$$

Proceeding as in (3.58) it is possible to find that

$$\|I_1\|_{L^1_T(C^\gamma)} \leq c\,(\|u_0\|_{H^{1+\gamma+s}}, T)\,,$$
$$\|I_2\|_{L^1_T(C^\gamma)} \leq c\,(\|u_0\|_{H^{1+\gamma+s}}, T)\,.$$

From the above estimates we finally conclude that

$$\|W \cdot \nabla^2 u\|_{L^1_T(C^\gamma)} \leq c\,(\|u_0\|_{H^{1+\gamma+s}}, T)\,.$$

□

Chapter 4

The Muskat problem with viscosity jump

4.1 Introduction

This chapter studies the dynamics of flows in porous media. This scenario is modeled using the classical Darcy's law [50]

$$\mu(x,t)u(x,t) = -\nabla p(x,t) - \rho(x,t)e_d, \tag{4.1}$$

where the velocity of the fluid u is proportional to the spatial gradient pressure ∇p and the gravity force. Above x is the space variable in $\mathbb{R}^d$ for $d = 2$, or 3, $t \geq 0$ is time and e_d is the last canonical vector. In the momentum equation, velocity replaces flow acceleration due to the porosity of the medium. It appears with the viscosity $\mu(x,t)$ divided by the permeability constant κ, here equal to one for simplicity of the exposition. The gravitational field comes with the density of the fluid $\rho(x,t)$ multiplied by the gravitational constant g, which is also normalized to one for clarity.

In this work the flow is incompressible

$$\nabla \cdot u(x,t) = 0, \tag{4.2}$$

and takes into consideration the dynamics of two immiscible fluids permeating the porous medium $\mathbb{R}^d$ with different constant densities and viscosities

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t). \end{cases} \tag{4.3}$$

The open sets $D^1(t)$ and $D^2(t)$ are connected with $\mathbb{R}^d = D^1(t) \cup D^2(t) \cup \partial D^j(t)$, $j = 1, 2$ and move with the velocity of the fluid

$$\frac{dx}{dt}(t) = u(x(t),t), \quad \forall\, x(t) \in D^j(t), \text{ or } x(t) \in \partial D^j(t). \tag{4.4}$$

The evolution equation above is well-defined at the free boundary even though the velocity is not continuous. The discontinuity in the velocity holds due to the density and viscosity jumps.

But what matters is the velocity in the normal direction, which is continuous thanks to the incompressibility of the velocity. The geometry of the problem is due to the gravitational force, with the fluid of viscosity μ^2 and density ρ^2 located mainly below the fluid of viscosity μ^1 and density ρ^1. In particular, there exists a constant $M > 1$ large enough such that $\mathbb{R}^{d-1} \times (-\infty, -M] \subset D^2(t)$.

We are then dealing with the well-established Muskat problem, whose main interest is about the dynamics of the free boundary $\partial D^j(t)$, especially between water and oil [97]. In this work, we study precisely this density-viscosity jump scenario, i.e. when there is a viscosity jump together with a density jump between the two fluids. Due to its wide applicability, this problem has been extensively studied [9]. In particular from the physical and experimental point of view, as in the two-dimensional case the phenomena is mathematically analogous to the two-phase Hele-Shaw cell evolution problem [104].

From the mathematical point of view, in the last decades there has been a lot of effort to understand the problem as it generates very interesting incompressible fluid dynamics behavior [61].

An important characteristic of the problem is that its Eulerian-Lagrangian formulation (4.1)-(4.4) understood in a weak sense provides an equivalent self-evolution equation for the interface $\partial D^j(t)$. This is the so-called contour evolution system, which we will now provide for 3D Muskat in order to understand the dynamics of its solutions.

Due to the irrotationality of the velocity in each domain $D^j(t)$, the vorticity is concentrated on the interface $\partial D^j(t)$. That is the vorticity is given by a delta distribution as follows

$$\nabla \wedge u(x,t) = \omega(\alpha,t)\delta(x = X(\alpha,t)),$$

where $\omega(\alpha,t)$ is the amplitude of the vorticity and $X(\alpha,t)$ is a global parameterization of $\partial D^j(t)$ with

$$\partial D^j(t) = \{X(\alpha,t) : \alpha \in \mathbb{R}^2\}.$$

It means that

$$\int_{\mathbb{R}^3} u(x,t) \cdot \nabla \wedge \varphi(x) dx = \int_{\mathbb{R}^2} \omega(\alpha,t) \cdot \varphi(X(\alpha,t)) d\alpha,$$

for any smooth compactly supported field φ. The evolution equation reads

$$\partial_t X(\alpha,t) = BR(X,\omega)(\alpha,t) + C_1(\alpha,t)\partial_{\alpha_1} X(\alpha,t) + C_2(\alpha,t)\partial_{\alpha_2} X(\alpha,t), \tag{4.5}$$

where BR is the well-known Birkhoff-Rott integral

$$BR(X,\omega)(\alpha,t) = -\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{X(\alpha,t) - X(\beta,t)}{|X(\alpha,t) - X(\beta,t)|^3} \wedge \omega(\beta,t) d\beta, \tag{4.6}$$

which appears using the Biot-Savart law and taking limits to the free boundary. Above the constants C_1 and C_2 represent the possible change of coordinates for the evolving surface and the prefix p.v. indicates a principal value integral. It is possible to close the system using that the velocity is given by different potentials in each domain and we denote the potential jump across the interface by the function $\Omega(\alpha,t)$. Taking limits approaching the free boundary in Darcy's law yields the non-local implicit identity

$$\Omega(\alpha,t) = A_\mu \mathcal{D}(\Omega)(\alpha,t) - 2A_\rho X_3(\alpha,t), \quad A_\mu = \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1}, \quad A_\rho = \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1}, \tag{4.7}$$

where $\mathcal{D}$ is the double layer potential

$$\mathcal{D}(\Omega)(\alpha,t) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{X(\alpha,t)-X(\beta,t)}{|X(\alpha,t)-X(\beta,t)|^3} \cdot \partial_{\alpha_1} X(\beta,t) \wedge \partial_{\alpha_2} X(\beta,t)\Omega(\beta,t)d\beta. \tag{4.8}$$

In that limit procedure, the continuity of the pressure at the free boundary is used, which is a consequence of the fact that Darcy's law (4.1) is understood in a weak sense. Relating the potential and the velocity jumps at the interface provides

$$\omega(\alpha,t) = \partial_{\alpha_2}\Omega(\alpha,t)\partial_{\alpha_1}X(\alpha,t) - \partial_{\alpha_1}\Omega(\alpha,t)\partial_{\alpha_2}X(\alpha,t), \tag{4.9}$$

and therefore it is possible to close the contour evolution system by (4.5)-(4.9) (see [37] for a detail derivation of the system).

Then the next question to ask is about the well-posedness of the problem. A remarkable peculiarity is that, in general, it does not hold. The system has to initially satisfy the so-called Rayleigh-Taylor condition (also called the Saffman-Taylor condition for the Muskat problem) to be well-posed. This condition holds if the difference of the gradient of the pressure in the normal direction at the interface is strictly positive [4],[5], i.e the stable regime. For large initial data, well-posedness was proved in [33] for the case with density jump in two and three dimensions. In that case, the Saffman-Taylor condition holds if the denser fluid lies below the less dense fluid. The density-viscosity jump stable situation was proved to exist locally in time in 2D [36] and in 3D [37]. Although these proofs use different approaches, it was essential in both proofs to find bounds for the amplitude of the vorticity in equation (4.7) in terms of the free boundary. There are recent results where local-in-time existence is shown in 2D for lower regular initial data given by graphs in the Sobolev space H^2 for the one-fluid case ($\mu^2 = 0$) [25] and in the two-fluid case ($\mu^2 \geq 0$) [95]. In the 2D density jump case the local-in-time existence has been shown for any subcritical Sobolev spaces $W^{2,p}$, $1 < p < \infty$ [31], and H^s, $3/2 < s < 2$ [94]. Here, the terminology subcritical is used in terms of the scaling of the problem, as $X^\lambda(\alpha,t) = \lambda^{-1}X(\lambda\alpha,\lambda t)$ and $\omega^\lambda(\alpha,t) = \omega(\lambda\alpha,\lambda t)$ are solution of (4.5)-(4.9) for any $\lambda \geq 0$ if $X(\alpha,t)$ and $\omega(\alpha,t)$ are. Therefore $\dot{W}^{1,\infty}$, $\dot{W}^{2,1}$ and $\dot{H}^{3/2}$ are critical and invariant homogeneous spaces for the system in 2D.

On the other hand, the Muskat problem can be unstable for some scenarios, when the Saffman-Taylor condition does not hold. In particular, if the difference of the gradient of the pressures in the normal direction at the interface is strictly negative, the contour evolution problem is ill-posed in the viscosity jump case [107] as well as the density jump situation [33] in subcritical spaces. With the Eulerian-Lagrangian formulation (4.1)-(4.4) it is possible to find weak solutions in the density jump case where the fluid densities mix in a strip close to the flat steady unstable state [110] and for any H^5 unstable graph [20]. In the contour dynamics setting, adding capillary forces to the system makes the contour equation well-posed [54]. When the Saffman-Taylor condition holds, the system is structurally stable to solutions without capillary forces if the surface tension coefficient is close to zero [6]. However, there exist unstable scenarios for interfaces interacting with capillary forces [99] which have been shown to have exponential growth for low order norms under small scales of times [68]. The system also exhibits finger shaped unstable stationary-states solutions [55].

A very important feature of this problem is that it can develop finite time singularities starting from stable situations. The Muskat problem then became the first incompressible model where blow-up for solutions with initial data in well-posed scenarios had been proven rigorously. Specifically, in the 2D density jump case, solutions starting in stable situation (denser fluid below a graph) become instantly analytic and move to unstable regimes in finite time [15]. In the unstable regime the interface is not a graph anymore, and at a later time the Muskat solution blows-up in finite time showing loss of regularity [16]. The geometry of those initial data are not well understood, as numerical experiments show that some solutions with large initial data can remain smooth [34], and the patterns can become more complicated for scenarios with fixed boundary effects [66]. As a matter of fact, some solutions can pass from the stable to the unstable regime and enter again to the stable regime [38].

The Muskat problem also develops a different kind of blow-up behavior in stable regimes: the so-called splash singularities. This singularity occurs if two different particles on the free boundary collide in finite time while the regularity of the interface is preserved. This collision cannot occur along a connected segment of the curve of particles in neither the density jump [35] nor the density-viscosity jump case [39]. In particular, the splash singularity is ruled out for the two-fluid case [60] but it takes place in one-fluid stable scenarios [20].

The question we study in this thesis is about the global in time existence, uniqueness, regularity and decay of solutions of the Muskat problem in stable regimes and ill-posedness in unstable regimes. In the viscosity [107], density [33] and density-viscosity jump 2D cases [55], [25] there exist global in time classical solutions for small initial data in subcritical norms which become instantly analytic, thereby demonstrating the parabolic character of the system in these situations. See [10] for the same result in the 2D density jump case with small initial data in critical norms, represented on the Fourier side by positive measure. In [31], global in time existence of classical solutions are shown to exist with small initial slope. In [29], global existence of 2D density-jump Muskat Lipschitz solutions are given for initial data with slope less than one. See [67] for an extension of the result with fixed boundary and [30] for the 3D scenario, where the L^∞ norm of the free boundary gradient has to be smaller than $1/3$. In [29] and [30] global existence and uniqueness is proved for solutions with continuous and bounded slope and L^1 in time bounded curvature in the density jump case for initial data in critical spaces with *medium* size. More specifically, the initial profiles are given by functions, i.e. $X(\alpha, 0) = (\alpha, f_0(\alpha))$, for a function $f_0(\alpha)$ of size less than k_0:

$$\|f_0\|_{\dot{\mathcal{F}}^{1,1}} = \int_{\mathbb{R}^{d-1}} d\xi |\xi| |\hat{f}_0(\xi)| < k_{0,d}, \qquad d = 2, 3,$$

where $k_{0,d}$ is an explicit constant, $k_{0,3} > 1/5$ in 3D and $k_{0,2} > 1/3$ in 2D. In [101], the optimal time decay of those solutions are proven, for initial data additionally bounded in subcritical Sobolev norms. We also point out work [13], where the Lipschitz solutions given in [16] are shown to become smooth by using a conditional regularity result given in [31] together with an instant generation of a modulus of continuity.

Next, we describe the main results and novelties in this work. This chapter extends the global existence results in 2D and 3D from [30] to the more general case with density-viscosity jump. Moreover, in 3D we improve the available constant for global existence and make it equal to the 2D constant in the $A_\mu = 0$ case. Precisely, it is given by initial data satisfying that

$$\|f_0\|_{\dot{\mathcal{F}}^{1,1}} = \int d\xi |\xi| |\hat{f}_0(\xi)| < k(|A_\mu|),$$

where $k : [0,1] \to [k(1), k_0]$ is decreasing and $k(0) = k_0 = k_{0,2}$. We would like to point out that due to the nature of equation (4.7), maximum principles are not available for the amplitude and the slopes in the L^∞ norm and the parabolic character of the equation is not as clear as in the case $A_\mu = 0$. We provide the first global existence result for this important scenario in a critical space. The space $\dot{\mathcal{F}}^{1,1}$ appears as a natural framework to perform the task of inverting the operator $(I - A_\mu \mathcal{D})$ in order to get bounds for ω in terms of the interface. In particular, we also improve the method in [30] as we are able to show smoothing effects, proving that solutions with medium size initial data become instantly analytic. Furthermore, we show uniform bounds of the interface in L^∞ and L^2 norms with analytic weights. Then, we show optimal decay rates for the analyticity of the critical solutions, improving the results in [101].

Finally, we show with the new approach that solutions are ill-posed in the unstable regime even for low regularity solutions understood in the contour dynamics setting. We give precise statements of these results in Section 4.3. In next section we provide the contour equations we use throughout the chapter.

4.2 Formulation of the Muskat Problem with Viscosity Jump

We start by deriving the contour equation formula given by (4.5)-(4.9) in terms of a graph. This equation will be used throughout the chapter to state the main results and to prove them. To simplify notation we shall write $f(\alpha, t) = f(\alpha)$ when there is no danger of confusion.

In the 3D case, if the evolving interface can be described as a graph

$$X(\alpha, t) = (\alpha_1, \alpha_2, f(\alpha, t)), \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2,$$

then the equations (4.5) are reduced to one equation as follows

$$0 = -\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{(\alpha_2 - \beta_2)\omega_3(\beta) - \omega_2(\beta)(f(\alpha) - f(\beta))}{|(\alpha, f(\alpha)) - (\beta, f(\beta))|^3} d\beta + C_1(\alpha),$$

$$0 = -\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{\omega_1(\beta)(f(\alpha) - f(\beta)) - (\alpha_1 - \beta_1)\omega_3(\beta)}{|(\alpha, f(\alpha)) - (\beta, f(\beta))|^3} d\beta + C_2(\alpha),$$

$$f_t(\alpha) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{(\alpha_1 - \beta_1)\omega_2(\beta) - (\alpha_2 - \beta_2)\omega_1(\beta)}{|(\alpha, f(\alpha)) - (\beta, f(\beta))|^3} d\beta + C_1(\alpha)\partial_{\alpha_1} f(\alpha) + C_2(\alpha)\partial_{\alpha_2} f(\alpha).$$

Thus, substituting the constants from the tangent terms into the evolution equation and applying a change of variables, we obtain the equation for f:

$$f_t(\alpha) = I_1(\alpha) + I_2(\alpha) + I_3(\alpha), \tag{4.10}$$

where

$$I_1(\alpha) = -\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{\beta_1\omega_2(\alpha - \beta) - \beta_2\omega_1(\alpha - \beta)}{(1 + (\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}} \frac{d\beta}{|\beta|^3}, \tag{4.11}$$

$$I_2(\alpha)=\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\Delta_\beta f(\alpha)\partial_{\alpha_2}f(\alpha)\omega_1(\alpha-\beta)-\Delta_\beta f(\alpha)\partial_{\alpha_1}f(\alpha)\omega_2(\alpha-\beta)}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}, \tag{4.12}$$

$$I_3(\alpha)=\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\beta_2\partial_{\alpha_1}f(\alpha)-\beta_1\partial_{\alpha_2}f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}}\omega_3(\alpha-\beta)\frac{d\beta}{|\beta|^3}. \tag{4.13}$$

Above we use the notation $\Delta_\beta f(\alpha)$ for

$$\Delta_\beta f(\alpha)=(f(\alpha)-f(\beta))|\beta|^{-1}.$$

We have the following equations for the vorticity coming from (4.9):

$$\omega_1=\partial_{\alpha_2}\Omega,\quad \omega_2=-\partial_{\alpha_1}\Omega,\quad \omega_3=\partial_{\alpha_2}\Omega\partial_{\alpha_1}f-\partial_{\alpha_1}\Omega\partial_{\alpha_2}f. \tag{4.14}$$

Introducing (4.14) into (4.11) and (4.12) they can be written as

$$I_1(\alpha)=\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{1}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}}\frac{\beta}{|\beta|^3}\cdot\nabla\Omega(\alpha-\beta)d\beta,$$

$$I_2(\alpha)=\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\Delta_\beta f(\alpha)\nabla f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}}\cdot\frac{\nabla\Omega(\alpha-\beta)}{|\beta|^2}d\beta. \tag{4.15}$$

By adding and subtracting the appropriate quantity, we obtain the following

$$I_1(\alpha)=\frac{1}{2}\Lambda\Omega(\alpha)+\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\Big((1+(\Delta_\beta f(\alpha))^2)^{-\frac{3}{2}}-1\Big)\frac{\beta}{|\beta|}\cdot\nabla\Omega(\alpha-\beta)d\beta,$$

where the operator Λ is given by the Riesz transforms

$$\Lambda=R_1\partial_{\alpha_1}+R_2\partial_{\alpha_2} \tag{4.16}$$

and also as a Fourier multiplier by $\widehat{\Lambda}=|\xi|$. Plugging the identity for Ω (4.7), the equation below shows the parabolic structure of the equation as

$$\begin{aligned}I_1(\alpha)=&-A_\rho\Lambda f(\alpha)+\frac{A_\mu}{2}\Lambda\mathcal{D}(\Omega)(\alpha)\\&+\frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\Big((1+(\Delta_\beta f(\alpha))^2)^{-\frac{3}{2}}-1\Big)\frac{\beta}{|\beta|}\cdot\nabla\Omega(\alpha-\beta)d\beta.\end{aligned} \tag{4.17}$$

Using formulas (4.14) and (4.7) in $I_3(\alpha)$ (4.13) we are able to find that

$$I_3(\alpha)=\frac{A_\mu}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\beta\cdot\nabla^\perp f(\alpha)\nabla\mathcal{D}(\Omega)(\alpha-\beta)\cdot\nabla^\perp f(\alpha-\beta)}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^3}. \tag{4.18}$$

We can finally write the contour equation (4.10) by using formulas (4.17), (4.15) and (4.18) to get:

$$f_t=-A_\rho\Lambda f+N(f),\qquad\text{where}\qquad N(f)=N_1(f)+N_2(f)+N_3(f), \tag{4.19}$$

where $N(f) = N(f,\Omega)$ and

$$
\begin{aligned}
N_1 &= \frac{A_\mu}{2}\Lambda\mathcal{D}(\Omega)(\alpha),\\
N_2 &= \frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\left(\frac{\frac{\beta}{|\beta|}+\Delta_\beta f(\alpha)\nabla f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}}-\frac{\beta}{|\beta|}\right)\cdot\frac{\nabla\Omega(\alpha-\beta)}{|\beta|^2}d\beta,\\
N_3 &= \frac{A_\mu}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\beta\cdot\nabla^\perp f(\alpha)\nabla\mathcal{D}(\Omega)(\alpha-\beta)\cdot\nabla^\perp f(\alpha-\beta)}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^3}.
\end{aligned}
\tag{4.20}
$$

The equation for Ω is given implicitly by

$$\Omega(\alpha,t) = A_\mu\mathcal{D}(\Omega)(\alpha,t) - 2A_\rho f(\alpha,t), \tag{4.21}$$

where the operator $\mathcal{D}(\Omega)$ (4.8) is rewritten as follows

$$\mathcal{D}(\Omega)(\alpha) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\Delta_\beta f(\alpha)-\frac{\beta\cdot\nabla f(\alpha-\beta)}{|\beta|}}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}}\frac{\Omega(\alpha-\beta)}{|\beta|^2}. \tag{4.22}$$

Note that the derivatives of $\mathcal{D}(\Omega)$ can be written in the following manner

$$\partial_{\alpha_i}\mathcal{D}(\Omega)(\alpha,t) = 2BR(f,\omega)(\alpha,t)\cdot\partial_{\alpha_i}(\alpha_1,\alpha_2,f(\alpha)), \tag{4.23}$$

and therefore

$$\partial_{\alpha_i}\Omega(\alpha,t) - 2A_\mu BR(f,\omega)(\alpha,t)\cdot\partial_{\alpha_i}(\alpha_1,\alpha_2,f(\alpha)) = -2A_\rho\partial_{\alpha_i}f(\alpha,t). \tag{4.24}$$

In the case of a graph, the Birkhoff-Rott integrals are also reduced in the following manner

$$BR(f,\omega) \stackrel{\text{def}}{=} (BR_1(f,\omega), BR_2(f,\omega), BR_3(f,\omega)),$$

where we use the shorthand $BR_i \stackrel{\text{def}}{=} BR_i(f,\omega)$ to be the terms

$$BR_1 = \frac{-1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\frac{\beta_2}{|\beta|}\omega_3(\alpha-\beta)-\Delta_\beta f(\alpha)\omega_2(\alpha-\beta)}{(1+\Delta_\beta f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}, \tag{4.25}$$

$$BR_2 = \frac{-1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\Delta_\beta f(\alpha)\omega_1(\alpha-\beta)-\frac{\beta_1}{|\beta|}\omega_3(\alpha-\beta)}{(1+\Delta_\beta f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}, \tag{4.26}$$

$$BR_3 = \frac{-1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2}\frac{\frac{\beta_1}{|\beta|}\omega_2(\alpha-\beta)-\frac{\beta_2}{|\beta|}\omega_1(\alpha-\beta)}{(1+\Delta_\beta f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}. \tag{4.27}$$

We state now the results in 2D. Proceeding similarly as above one obtains that

$$f_t = -A_\rho\Lambda f + N(f), \qquad \text{where} \qquad N(f) = N_1(f) + N_2(f), \tag{4.28}$$

where $N(f) = N(f,\Omega)$ and

$$\begin{aligned} N_1 &= \frac{A_\mu}{2}\Lambda\mathcal{D}(\Omega)(\alpha), \\ N_2 &= \frac{1}{2\pi}\text{p.v.}\int \frac{\Delta_\beta f(\alpha) - \partial_\alpha f(\alpha)}{1+\Delta_\beta f(\alpha)^2}\Delta_\beta f(\alpha)\frac{\partial_\alpha\Omega(\alpha-\beta)}{\beta}d\beta. \end{aligned} \tag{4.29}$$

The equation for Ω is given implicitly by

$$\Omega(\alpha,t) = A_\mu\mathcal{D}(\Omega)(\alpha,t) - 2A_\rho f(\alpha,t), \tag{4.30}$$

where the operator $\mathcal{D}(\Omega)(\alpha,t)$ is rewritten as follows

$$\mathcal{D}(\Omega)(\alpha,t) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\Delta_\beta f(\alpha) - \partial_\alpha f(\alpha-\beta)}{1+\Delta_\beta f(\alpha)^2}\frac{\Omega(\alpha-\beta)}{\beta}d\beta. \tag{4.31}$$

Note that the vorticity is given by $\omega(\alpha) = \partial_\alpha\Omega(\alpha)$.

4.3 Main Results

We present the main results and briefly give an outline of the structure of this chapter. The first result is global well-posedness in the critical space $\dot{\mathcal{F}}^{1,1}\cap L^2$ in 3D, where we define the norms

$$\|f\|_{\dot{\mathcal{F}}^{s,1}} \stackrel{\text{def}}{=} \int |\xi|^s|\hat{f}(\xi)|d\xi, \quad s > -2.$$

We also denote $\|f\|_{\dot{\mathcal{F}}^{0,1}} = \|f\|_{\mathcal{F}^{0,1}}$.

Theorem 4.3.1 (Existence and Uniqueness, 3D). *Let $f_0 \in \dot{\mathcal{F}}^{1,1}\cap L^2$ satisfy the bound*

$$\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k(|A_\mu|)$$

for a constant $k(|A_\mu|)$ depending on the Atwood number A_μ. Then there exists a unique solution to (4.19-4.22) with $f \in L^\infty(0,T;\dot{\mathcal{F}}^{1,1}\cap L^2)\cap L^1(0,T;\dot{\mathcal{F}}^{2,1})$ such that $f(\alpha,0) = f_0(\alpha)$ and

$$\|f\|_{L^2}(t) \le \|f_0\|_{L^2}, \quad \|f\|_{\dot{\mathcal{F}}^{1,1}}(t) + \sigma\int_0^t \|f\|_{\dot{\mathcal{F}}^{2,1}}(\tau)d\tau \le \|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k(|A_\mu|), \tag{4.32}$$

for a positive constant σ depending on the initial profile $f_0(\alpha)$.

In the 2D case, we analogously have the following:

Theorem 4.3.2 (Existence and Uniqueness in 2D). *Let $f_0 \in \dot{\mathcal{F}}^{1,1}\cap L^2$ satisfy the bound*

$$\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < c(|A_\mu|)$$

for a constant $c(|A_\mu|)$ depending on the Atwood number A_μ. Then there exists a unique solution to (4.28-4.31) with $f \in L^\infty(0,T;\dot{\mathcal{F}}^{1,1}\cap L^2)\cap L^1(0,T;\dot{\mathcal{F}}^{2,1})$ such that $f(\alpha,0) = f_0(\alpha)$ and

$$\|f\|_{L^2}(t) \le \|f_0\|_{L^2}, \quad \|f\|_{\dot{\mathcal{F}}^{1,1}}(t) + \sigma\int_0^t \|f\|_{\dot{\mathcal{F}}^{2,1}}(\tau)d\tau \le \|f_0\|_{\dot{\mathcal{F}}^{1,1}} < c(|A_\mu|),$$

for a positive constant σ depending on the initial profile $f_0(\alpha)$,

$$\sigma(\|f_0\|_{\dot{\mathcal{F}}^{1,1}}) = -1 + \nu + \frac{2\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}\left(3 - \|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}\right)}{\left(1 - \|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}\right)^2}$$
$$+ A_\mu \frac{2\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\left(2A_\mu\|f_0\|^5_{\dot{\mathcal{F}}^{1,1}} - 6\|f_0\|^4_{\dot{\mathcal{F}}^{1,1}} - 8A_\mu\|f_0\|^3_{\dot{\mathcal{F}}^{1,1}} + 4\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}} - 2A_\mu\|f_0\|_{\dot{\mathcal{F}}^{1,1}} + 2\right)}{\left(1 - \|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}\right)^2\left(1 - \|f_0\|^2_{\dot{\mathcal{F}}^{1,1}} - 2A_\mu\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\right)^2}.$$

Computing the constant explicitly for $|A_\mu| = 1$, we obtain $c(1) \approx 0.128267$.

As noted in the introductory section, in the 3D setting, when $A_\mu = 0$, the constant $k(0)$ matches the size of the initial data in the 2D without viscosity jump proven in [29], and therefore, improves the size of the initial data in the 3D case without viscosity jump given by [30].

In the graph below the 3D constant $k(|A_\mu|)$ is pictured with respect to $|A_\mu|$. The maximum is $k_0 \approx 0.362606$ and the minimum $k(1) \approx 0.080604$. The graph in the Figure arises from estimating the size of initial data, $k(|A_\mu|)$, needed to satisfy the positivity condition (4.60) of the high order rational polynomial given in the proof.

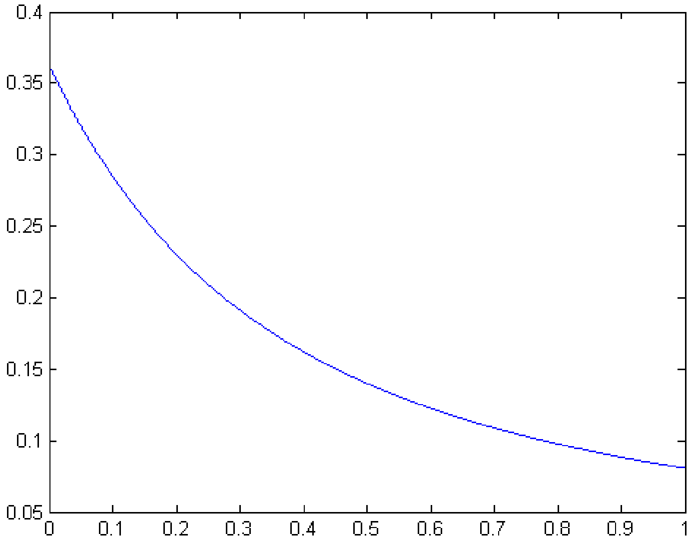

Figure 4.1: $k(|A_\mu|)$

To prove Theorem 4.3.1 and in particular (4.32), we first need to prove apriori estimates on the vorticity and potential jump functions. These estimates on $\|\omega_i\|_{\dot{\mathcal{F}}^{s,1}}$ for $s = 0, 1$ are computed in Section 4.4. The key point of the vorticity estimates is to demonstrate a bound on $\|\omega_i\|_{\dot{\mathcal{F}}^{s,1}}$ by a constant multiple of $\|f\|_{\dot{\mathcal{F}}^{s+1,1}}$, as $\omega_i(\alpha)$ is of similar regularity to $\nabla f(\alpha)$.

Next, we introduce the following norms with analytic weights:

$$\|f\|^p_{\dot{\mathcal{F}}^{s,p}_\nu}(t) \stackrel{\text{def}}{=} \int |\xi|^{sp} e^{p\nu t|\xi|} |\hat{f}(\xi,t)|^p d\xi, \quad \nu > 0, \quad s \geq 0, \quad p \geq 1, \tag{4.33}$$

with $\|f\|_{\dot{\mathcal{F}}^{0,p}_\nu} = \|f\|_{\mathcal{F}^{0,p}_\nu}$. In the Section 4.5, we then use the vorticity estimates to prove uniform bounds on the analytic weighted quantity $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t)$:

Theorem 4.3.3 (Instant Analyticity)**.** *Suppose $f(\alpha, t)$ is a solution to (4.19-4.22) with initial data satisfying $\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k(|A_\mu|)$ or (4.28-4.31) with initial data satisfying $\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < c(|A_\mu|)$. Then there exist $\nu = \nu(\|f_0\|_{\dot{\mathcal{F}}^{1,1}}) > 0$ such that the evolution of the quantity $\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}$ satisfies the estimate*

$$\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \sigma \int_0^t \|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau) d\tau \leq \|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k(|A_\mu|). \tag{4.34}$$

Furthermore, if $f_0 \in L^2$ then

$$\|f\|_{\mathcal{F}^{0,2}_\nu} \leq \|f_0\|_{L^2} \exp(R(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})), \tag{4.35}$$

where R is a positive rational polynomial.

Setting $\nu = 0$, we obtain the desired estimate (4.32). Following the instant analyticity argument, we present an L^2 maximum principle for the Muskat problem with viscosity jump in Section 4.6. Next, in Section 4.7, we give an argument for uniqueness of solutions in the space $\mathcal{F}^{0,1}$, noting that $\mathcal{F}^{0,1} \hookrightarrow L^\infty$. All of these apriori estimates finally allow us to perform a regularization argument in Section 4.8. In this argument, we perform an appropriate mollification of the interface evolution equation for $f(\alpha, t)$ and show that the regularized solutions $f^{\varepsilon_n}(\alpha, t)$ converge strongly to $f(\alpha, t)$ in $L^2(0, T; \dot{\mathcal{F}}^{1,1})$ and satisfy (4.32). Taking the limit $f^{\varepsilon_n}(\alpha, t) \longrightarrow f(\alpha, t)$, we establish the global wellposedness result of Theorem 4.3.1.

In this thesis, we also show analytic results in L^2 spaces in Section 4.9. Specifically, we prove uniform bounds on an analytic L^2 norm, as given by (4.35) as well as

$$\frac{d}{dt}\|f\|^2_{\dot{\mathcal{F}}^{s,2}_\nu}(t) \leq -\sigma \|f\|^2_{\dot{\mathcal{F}}^{s+1/2,2}_\nu} \tag{4.36}$$

for $1/2 \leq s \leq 3/2$. Note that in general, we denote $\dot{\mathcal{F}}^{s,2}_\nu$ by $\dot{H}^s_\nu$ for $s \neq 0$ and $\mathcal{F}^{0,2}_\nu$ by L^2_ν throughout the chapter. We will use this L^2 estimates to show the L^2 decay and ill-posedness results.

Given solutions with initial data as described in Theorem 4.3.1, in Section 4.10 we obtain large-time decay for solutions to the Muskat problem by using estimates similar to (4.34), (4.35) and (4.36):

Theorem 4.3.4 (Sharp Decay Estimates)**.** *Suppose $f(\alpha, t)$ is a solution to (4.19-4.22) with initial data satisfying $\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k(|A_\mu|)$ and $\|f_0\|_{L^2} < \infty$. Then for any $0 \leq s \leq 1$*

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}(t) \leq C_s(1+t)^{-s-1+\lambda},$$

for arbitrarily small $\lambda > 0$ and some nonnegative constant C_s depending on the initial profile $f_0(\alpha)$ and the exponent s. Moreover, for any $T > 0$, there exists a constant $C_{T,s}$ depending on f_0, T and s such that

$$\|f\|_{\dot{H}^s_\nu}(t) \le C_{T,s} t^{-s},$$

for $t > T$. In 2D, we have the following decay rate for solutions with initial data satisfying $\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < c(|A_\mu|)$ and $\|f_0\|_{L^2}$:

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}(t) \le C_s(1+t)^{-s-1/2+\lambda}.$$

The H^s_ν decay rates in 2D are the same.

Remark 4.3.5. *We call the decay rates in Theorem 4.3.4 sharp for the following reason. If $f_0 \in \dot{\mathcal{F}}^{1,1} \cap L^2$, then it can be seen that $f_0 \in \dot{\mathcal{F}}^{s',1}$ for $-1 < s' \le 1$ but $f_0(\alpha)$ need not be in $\dot{\mathcal{F}}^{-1,1}$. If we consider the linearized contour equation with initial data $\|f_0\|_{\dot{\mathcal{F}}^{s',1}}$ for $-1 < s' < 1$, then for any $s > s'$, we have the equivalence for the linear solutions*

$$\|f_0\|_{\dot{\mathcal{F}}^{s',1}} \approx \|t^{s-s'} \|e^{t\Lambda} f_0\|_{\dot{\mathcal{F}}^{s,1}}\|_{L^\infty_t}.$$

This estimate yields, for example, the optimal rate of $t^{s'-1}$ for decay of $\|f\|_{\dot{\mathcal{F}}^{1,1}}$. Because we at most can guarantee that $\|f_0\|_{\dot{\mathcal{F}}^{s',1}} < +\infty$ for $-1 < s' \le 1$ but not for any lower value of s, the decay rates above are sharp. Finally, for $\nu \neq 0$, since $\|f\|_{\dot{\mathcal{F}}^{s,1}} \le \|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}$, the rates are also sharp for the analytic weighted norms.

In Section 4.10, we additionally note that for f_0 satisfying the conditions of Theorem 4.3.4, it immediately follows that the solution $f(\alpha, t)$ is in the spaces $\dot{\mathcal{F}}^{s,1} \cap \dot{H}^{s'}$ for any $s > -1$ and $s' \ge 0$. Moreover, due to the decay of the quantity $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, we can show that there exists a constant k_s and time T_s depending on $s > 1$ and the initial profile f_0 such that

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}(t) + \sigma_s \int_{T_s}^t \|f\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}(\tau) d\tau \le k_s \tag{4.37}$$

for some $\nu > 0$ and for all $t > T_s$ for a time T_s large enough and depending on s and f_0. Therefore, we obtain decay rates for $t > T_s$:

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}(t) \le C_s t^{-s-1+\lambda}$$

analogously to Theorem 4.3.4. We can draw similar conclusions for the Sobolev norms with analytic weight.

Finally, and importantly, we use the L^2_ν uniform bound (4.35) to obtain an ill-posedness argument for the unstable regime of the Muskat problem in Section 4.11:

Theorem 4.3.6 (Ill-posedness)**.** *For every $s > 0$ and $\epsilon > 0$, there exist a solution $\tilde{f}$ to the unstable regime and $0 < \delta < \epsilon$ such that $\|\tilde{f}\|_{H^s}(0) < \epsilon$ but $\|\tilde{f}\|_{H^s}(\delta) = \infty$.*

This ill-posedness result is very significant because we show instantaneous blow-up of solutions in very low regularity spaces.

4.4 A Priori Estimates on ω

In this section, we will prove the necessary estimates on $\|\omega_i\|_{\dot{\mathcal{F}}^{s,1}}$ for $s = 0, 1$ and $i = 1, 2, 3$. These estimates will be used later to prove the bound (4.34) on the evolution of a solution in $\|f\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}$. We first show that $\|\omega_i\|_{\mathcal{F}^{0,1}}$ is bounded by quantities depending only on the characteristics of the fluids and $\|f\|_{\dot{\mathcal{F}}^{1,1}}$. Then, using the estimates on $\|\omega_i\|_{\mathcal{F}^{0,1}}$, we further show that the quantities $\|\omega_i\|_{\dot{\mathcal{F}}^{1,1}}$ for $i = 1, 2, 3$ are linearly bounded by $\|f\|_{\dot{\mathcal{F}}^{2,1}}$ with the linear constant depending on $\|f\|_{\dot{\mathcal{F}}^{1,1}}$.

Proposition 4.4.1. *Given the constants S_1, C_1, C_2 depending on A_μ, $\|f\|_{\dot{\mathcal{F}}^{1,1}}$ that are defined by*

$$S_1 = \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}}, \qquad C_1 = \frac{1 - A_\mu S_1}{1 - 5A_\mu S_1}, \qquad C_2 = \frac{C_1}{(1 - 2A_\mu S_1)(1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}})}, \tag{4.38}$$

we have the following estimates

$$\|\omega_1\|_{\mathcal{F}^{0,1}} = \|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} \le 2C_1 A_\rho \|f\|_{\dot{\mathcal{F}}^{1,1}}, \tag{4.39}$$

$$\|\omega_2\|_{\mathcal{F}^{0,1}} = \|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} \le 2C_1 A_\rho \|f\|_{\dot{\mathcal{F}}^{1,1}}, \tag{4.40}$$

and

$$\begin{aligned} \|\omega_3\|_{\mathcal{F}^{0,1}} &\le 12 A_\mu A_\rho C_2 \|f\|^3_{\dot{\mathcal{F}}^{1,1}}, \\ \|\partial_{\alpha_i}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\le 6A_\rho C_2 \|f\|^2_{\dot{\mathcal{F}}^{1,1}}, \quad i = 1, 2. \end{aligned} \tag{4.41}$$

For the potential jump function Ω, we moreover have the estimate

$$\|\Omega\|_{\dot{\mathcal{F}}^{1,1}} \le 2A_\rho B_1 \|f\|_{\dot{\mathcal{F}}^{1,1}}, \tag{4.42}$$

where

$$B_1 = \frac{1 - 2A_\mu S_1}{1 - 8A_\mu S_1}. \tag{4.43}$$

Proof. First, by formulas (4.14) and (4.21) we have that

$$\|\omega_1\|_{\mathcal{F}^{0,1}} = \|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}}, \qquad \|\omega_1\|_{\mathcal{F}^{0,1}} = \|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}}, \tag{4.44}$$

and

$$\begin{aligned} \|\omega_3\|_{\mathcal{F}^{0,1}} &= \|\partial_{\alpha_2}\Omega\partial_{\alpha_1} f - \partial_{\alpha_1}\Omega\partial_{\alpha_2} f\|_{\mathcal{F}^{0,1}} = A_\mu \|\partial_{\alpha_2}\mathcal{D}\partial_{\alpha_1} f - \partial_{\alpha_1}\mathcal{D}\partial_{\alpha_2} f\|_{\mathcal{F}^{0,1}} \\ &\le A_\mu \|f\|_{\dot{\mathcal{F}}^{1,1}} \left(\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}} \right), \end{aligned} \tag{4.45}$$

so it suffices to bound the quantities $\|\partial_{\alpha_i}\Omega\|_{\mathcal{F}^{0,1}}$ and $\|\partial_{\alpha_i}\mathcal{D}\|_{\mathcal{F}^{0,1}}$ for $i = 1, 2$. Notice that from (4.21) and (4.23) we have that

$$\begin{aligned} \|\partial_{\alpha_i}\Omega\|_{\mathcal{F}^{0,1}} &\le A_\mu \|\partial_{\alpha_i}\mathcal{D}\|_{\mathcal{F}^{0,1}} + 2A_\rho \|f\|_{\dot{\mathcal{F}}^{1,1}}, \\ \|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\le 2\|BR_1\|_{\mathcal{F}^{0,1}} + 2\|BR_3\partial_{\alpha_1} f\|_{\mathcal{F}^{0,1}}, \\ \|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\le 2\|BR_2\|_{\mathcal{F}^{0,1}} + 2\|BR_3\partial_{\alpha_2} f\|_{\mathcal{F}^{0,1}}. \end{aligned} \tag{4.46}$$

Thus, we proceed to bound the terms $\|BR_1\|_{\mathcal{F}^{0,1}}$, $\|BR_2\|_{\mathcal{F}^{0,1}}$ and $\|BR_3\|_{\mathcal{F}^{0,1}}$, given by (4.25), (4.26) and (4.27). We start with the term $\|BR_1\|_{\mathcal{F}^{0,1}}$. We first need to bound the Fourier transform of the Birkhoff-Rott integral terms. For the first term in BR_1,

$$BR_{11}(f)(\alpha) = \frac{-1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{\frac{\beta_2}{|\beta|}\omega_3(\alpha-\beta)}{(1+\Delta_\beta f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}, \tag{4.47}$$

we first apply a change of variables in β.

$$BR_{11}(f)(\alpha) = \frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{\frac{\beta_2}{|\beta|}\omega_3(\alpha+\beta)}{(1+\Delta_{-\beta} f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}.$$

Hence,

$$BR_{11}(f)(\alpha) = \frac{-1}{8\pi}\Big(\text{p.v.}\int_{\mathbb{R}^2} \frac{\frac{\beta_2}{|\beta|}\omega_3(\alpha-\beta)}{(1+\Delta_\beta f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2} - \text{p.v.}\int_{\mathbb{R}^2} \frac{\frac{\beta_2}{|\beta|}\omega_3(\alpha+\beta)}{(1+\Delta_{-\beta} f(\alpha)^2)^{\frac{3}{2}}}\frac{d\beta}{|\beta|^2}\Big).$$

By using the Taylor series expansion for the denominator, given by

$$\frac{1}{(1+x^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} a_n(-1)^n x^{2n}, \quad \text{where} \quad a_n = \frac{(2n+1)!}{(2^n n!)^2},$$

we obtain that

$$\begin{aligned}BR_{11}(f)(\alpha) = \frac{-1}{8\pi}\sum_{n\geq 0}(-1)^n a_n \int_{\mathbb{R}^2}\Big(&\frac{\beta_2}{|\beta|}\omega_3(\alpha-\beta)\Delta_\beta f(\alpha)^{2n}\\ &- \frac{\beta_2}{|\beta|}\omega_3(\alpha+\beta)\Delta_{-\beta} f(\alpha)^{2n}\Big)\frac{d\beta}{|\beta|^2}.\end{aligned}$$

Applying the Fourier transform, the products are transformed to convolutions:

$$\begin{aligned}\widehat{BR_{11}}(\xi) = \frac{-1}{8\pi}\sum_{n\geq 0}(-1)^n a_n \int_{\mathbb{R}^2}\frac{\beta_2}{|\beta|}\Big(&\hat{\omega_3}(\xi)e^{-i\beta\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,\beta)\big)\\ &- \hat{\omega_3}(\xi)e^{i\beta\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,-\beta)\big)\Big)\frac{d\beta}{|\beta|^2},\end{aligned}$$

where

$$m(\xi,\beta) = \frac{1-e^{-i\beta\cdot\xi}}{|\beta|}.$$

Writing the integral in polar coordinates with $\beta = ru$ and $u = (\cos(\theta), \sin(\theta))$,

$$\begin{aligned}\widehat{BR_{11}}(\xi) = \frac{-1}{8\pi}\sum_{n\geq 0}(-1)^n a_n \int_{-\pi}^{\pi}\int_0^{\infty}\sin(\theta)\Big(&\hat{\omega_3}(\xi)e^{-iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,r,u)\big)\\ &- \hat{\omega_3}(\xi)e^{iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,r,-u)\big)\Big)\frac{dr}{r}d\theta.\end{aligned}$$

By a change of variables in the radial variable,

$$\widehat{BR_{11}}(\xi) = \frac{-1}{8\pi}\sum_{n\geq 0}(-1)^n a_n \int_{-\pi}^{\pi}\int_0^{-\infty} \sin(\theta)\Big(\hat{\omega_3}(\xi)e^{iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,-r,u)\big) \\ - \hat{\omega_3}(\xi)e^{-iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,-r,-u)\big)\Big)\frac{-dr}{-r}d\theta.$$

Note that $m(\xi,-r,u) = -m(\xi,r,-u)$, and hence, we obtain

$$\widehat{BR_{11}}(\xi) = \frac{-1}{8\pi}\sum_{n\geq 0}(-1)^n a_n \int_{-\pi}^{\pi}\int_{-\infty}^{0} \sin(\theta)\Big(\hat{\omega_3}(\xi)e^{-iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,r,u)\big) \\ - \hat{\omega_3}(\xi)e^{iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,r,-u)\big)\Big)\frac{dr}{r}d\theta.$$

Thus,

$$\widehat{BR_{11}}(\xi) = \frac{-1}{16\pi}\sum_{n\geq 0}(-1)^n a_n \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \sin(\theta)\Big(\hat{\omega_3}(\xi)e^{-iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,r,u)\big) \\ - \hat{\omega_3}(\xi)e^{iru\cdot\xi} * \big(*^{2n}\, \hat{f}(\xi)m(\xi,r,-u)\big)\Big)\frac{dr}{r}d\theta.$$

Writing out of the convolutions in integral form and using the equality

$$m(\xi,r,u) = iu\cdot\xi\int_0^1 e^{-ir(1-s)u\cdot\xi}ds,$$

we obtain that

$$\widehat{BR_{11}}(\xi) \\ = \frac{-1}{16\pi}\sum_{n\geq 0}a_n\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}\sin(\theta)\hat{\omega_3}(\xi-\xi_1)\prod_{j=1}^{2n-1}(u\cdot(\xi_j-\xi_{j+1}))(u\cdot\xi_{2n})\hat{f}(\xi_j-\xi_{j+1}) \\ \int_0^1\cdots\int_0^1\int_{-\infty}^{\infty}\Big(e^{-iAr}-e^{iAr}\Big)\frac{dr}{r}ds_1\cdots ds_{2n}d\theta d\xi_1\cdots d\xi_{2n+1}.$$

where

$$A = u\cdot(\xi-\xi_1) + \sum_{j=1}^{2n-1}(1-s_j)u\cdot(\xi_j-\xi_{j+1}) + u\cdot\xi_{2n}.$$

Next, notice that

$$\Big|\int_0^1\cdots\int_0^1\int_{-\infty}^{\infty}\Big(e^{-iAr}-e^{iAr}\Big)\frac{dr}{r}ds_1\cdots ds_{2n}\Big| \\ \leq \pi\Big|\int_0^1\cdots\int_0^1 sgn(A)-sgn(-A)ds_1\cdots ds_{2n}\Big| \leq 2\pi.$$

Moreover, if $\xi = (\xi^{(1)}, \xi^{(2)})$, then

$$|u \cdot \xi| = |\cos(\theta)\xi^{(1)} + \sin(\theta)\xi^{(2)}| = |\xi||\sin(\theta + \alpha)|,$$

where α satisfies $\sin(\alpha) = \xi^{(1)}/|\xi|$, and therefore, $\cos(\alpha) = \xi^{(2)}/|\xi|$. Using these estimates,

$$|\widehat{BR_{11}}|(\xi) \leq \frac{1}{8}\sum_{n\geq 0} a_n \Big((*^{2n}|\cdot||\hat{f}(\cdot)|) * |\hat{\omega_3}(\cdot)|\Big)(\xi) \int_{-\pi}^{\pi} |\sin(\theta)| \prod_{j=1}^{2n} |\sin(\theta + \alpha_j)| d\theta$$

for some angles α_j. Finally, note that

$$\int_{-\pi}^{\pi} |\sin(\theta)| \prod_{j=1}^{2n} |\sin(\theta + \alpha_j)| d\theta \leq \int_{-\pi}^{\pi} |\sin(\theta)|^{2n+1} d\theta = 4/a_n.$$

Summarizing,

$$|\widehat{BR_{11}}|(\xi) \leq \frac{1}{2}\sum_{n\geq 0} \Big((*^{2n}|\cdot||\hat{f}(\cdot)|) * |\widehat{\omega}_3(\cdot)|\Big)(\xi). \tag{4.48}$$

The estimates on BR_{12}, BR_2 and BR_3 follow as the one on BR_{11}. We conclude that

$$\|BR_1\|_{\mathcal{F}^{0,1}} \leq \frac{1}{2}\frac{1}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(\|f\|_{\dot{\mathcal{F}}^{1,1}}\|\omega_2\|_{\mathcal{F}^{0,1}} + \|\omega_3\|_{\mathcal{F}^{0,1}}\right),$$
$$\|BR_2\|_{\mathcal{F}^{0,1}} \leq \frac{1}{2}\frac{1}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(\|f\|_{\dot{\mathcal{F}}^{1,1}}\|\omega_1\|_{\mathcal{F}^{0,1}} + \|\omega_3\|_{\mathcal{F}^{0,1}}\right),$$
$$\|BR_3\|_{\mathcal{F}^{0,1}} \leq \frac{1}{2}\frac{1}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(\|\omega_1\|_{\mathcal{F}^{0,1}} + \|\omega_2\|_{\mathcal{F}^{0,1}}\right),$$

Introducing this bounds into (4.46) we find that

$$\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} \leq \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(2\|\omega_2\|_{\mathcal{F}^{0,1}} + \|\omega_1\|_{\mathcal{F}^{0,1}}\right) + \frac{1}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}}\|\omega_3\|_{\mathcal{F}^{0,1}}.$$

Substituting the bounds for the vorticity (4.44),(4.45), it follows that

$$\begin{aligned}\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} \leq{}& \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(2\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}}\right) \\ &+ A_\mu \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}}\right).\end{aligned}$$

Analogously,

$$\begin{aligned}\|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}} \leq{}& \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(2\|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}}\right) \\ &+ A_\mu \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}} \left(\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}}\right).\end{aligned}$$

If we denote

$$S_1 = \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1 - \|f\|^2_{\dot{\mathcal{F}}^{1,1}}}, \qquad S_2 = \frac{S_1}{1 - A_\mu S_1}, \tag{4.49}$$

the above inequalities can be written as

$$\begin{aligned} \|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\leq S_2\left(2\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} + A_\mu\|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}}\right), \\ \|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\leq S_2\left(2\|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} + A_\mu\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}}\right). \end{aligned} \tag{4.50}$$

Therefore, it is not hard to see that

$$\begin{aligned} \|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\leq \frac{S_2(2 + A_\mu S_2)}{1 - (A_\mu S_2)^2}\Big(\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}}\Big), \\ \|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\leq \frac{S_2(2 + A_\mu S_2)}{1 - (A_\mu S_2)^2}\Big(\|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} + \|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}}\Big). \end{aligned} \tag{4.51}$$

By defining the following constants

$$c_1 = \frac{S_2}{1 - (A_\mu S_2)^2}(2 + A_\mu S_2), \qquad c_2 = \frac{S_2}{1 - (A_\mu S_2)^2}(1 + 2A_\mu S_2),$$

and recalling (4.46) and the bounds above we have that

$$\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} \leq A_\mu c_1\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} + A_\mu c_2\|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} + 2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}},$$

$$\|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} \leq A_\mu c_1\|\partial_{\alpha_2}\Omega\|_{\mathcal{F}^{0,1}} + A_\mu c_2\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}} + 2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}.$$

Therefore, we can conclude that

$$\|\partial_{\alpha_i}\Omega\|_{\mathcal{F}^{0,1}} \leq 2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}\frac{1}{1 - A_\mu(c_1 + c_2)}.$$

This expression can be simplified further to obtain that

$$\|\partial_{\alpha_i}\Omega\|_{\mathcal{F}^{0,1}} \leq 2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}\frac{1 - A_\mu S_1}{1 - 5A_\mu S_1}.$$

Going back to (4.51) we find that

$$\begin{aligned} \|\partial_{\alpha_i}\mathcal{D}\|_{\mathcal{F}^{0,1}} &\leq \frac{S_2}{1 - (A_\mu S_2)^2}3(1 + A_\mu S_2)2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}\frac{1 - A_\mu S_1}{1 - 5A_\mu S_1} \\ &= 6A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}\frac{S_1(1 - A_\mu S_1)}{(1 - 2A_\mu S_1)(1 - 5A_\mu S_1)} \end{aligned}$$

This last two bounds combined with the estimates (4.44), (4.45) conclude the proof of (4.39)-(4.41). Finally, to show (4.42), we do the following using (4.51):

$$\begin{aligned} \|\Omega\|_{\dot{\mathcal{F}}^{1,1}} &\leq A_\mu\|\partial_{\alpha_1}\mathcal{D}(\Omega)\|_{\mathcal{F}^{0,1}} + A_\mu\|\partial_{\alpha_2}\mathcal{D}(\Omega)\|_{\mathcal{F}^{0,1}} + 2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}} \\ &\leq \frac{6A_\mu S_2}{1 - A_\mu S_2}\|\Omega\|_{\dot{\mathcal{F}}^{1,1}} + 2A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}. \end{aligned}$$

Therefore,

$$\|\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq 2A_\rho\Big(\frac{1-2A_\mu S_1}{1-8A_\mu S_1}\Big)\|f\|_{\dot{\mathcal{F}}^{1,1}}.$$

This concludes the proof. □

Proposition 4.4.2. *Define the constants C_3, C_4 and C_5 depending on A_μ and $\|f\|_{\dot{\mathcal{F}}^{1,1}}$,*

$$\begin{aligned} C_3 &= \frac{1+\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}{1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}\left(1+A_\mu\frac{6\|f\|_{\dot{\mathcal{F}}^{1,1}}(1-A_\mu S_1)}{(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}})(1-2A_\mu S_1)(1-5A_\mu S_1)}\right),\\ C_4 &= \frac{1+S_2^2A_\mu^2\left(C_3+C_1+4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)}{1-3A_\mu S_2(1+A_\mu S_2)},\\ C_5 &= \frac{S_2}{\|f\|_{\dot{\mathcal{F}}^{1,1}}}\frac{3+A_\mu S_2(3+C_3+C_1+4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}})}{1-3A_\mu S_2(1+A_\mu S_2)}, \end{aligned} \tag{4.52}$$

where C_1, S_1 and S_2 are given by (4.38) *and* (4.49)*. Then, we have the following estimates*

$$\begin{aligned} \|\omega_1\|_{\dot{\mathcal{F}}^{1,1}} &= \|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq 2A_\rho C_4\|f\|_{\dot{\mathcal{F}}^{2,1}},\\ \|\omega_2\|_{\dot{\mathcal{F}}^{1,1}} &= \|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq 2A_\rho C_4\|f\|_{\dot{\mathcal{F}}^{2,1}}, \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} \|\omega_3\|_{\dot{\mathcal{F}}^{1,1}} &\leq 4A_\mu A_\rho\|f\|^2_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}}(C_5+3C_2),\\ \|\partial_{\alpha_i}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} &\leq 2A_\rho C_5\|f\|_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}}, \quad i=1,2. \end{aligned} \tag{4.54}$$

Moreover,

$$\|\Omega\|_{\dot{\mathcal{F}}^{2,1}} \leq 2A_\rho B_2\|f\|_{\dot{\mathcal{F}}^{2,1}},$$

where

$$B_2 = \frac{1+2S_2^2A_\mu(C_1+C_3+4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}})}{1-6A_\mu S_2(1+A_\mu S_2)}. \tag{4.55}$$

Proof. Using the formulas for the vorticity it follows that

$$\|\omega_1\|_{\dot{\mathcal{F}}^{1,1}} = \|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}}, \qquad \|\omega_2\|_{\dot{\mathcal{F}}^{1,1}} = \|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}},$$

$$\begin{aligned} \|\omega_3\|_{\dot{\mathcal{F}}^{1,1}} \leq{}& A_\mu\|f\|_{\dot{\mathcal{F}}^{1,1}}(\|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}})\\ &+A_\mu\|f\|_{\dot{\mathcal{F}}^{2,1}}(\|\partial_{\alpha_1}\mathcal{D}\|_{\mathcal{F}^{0,1}}+\|\partial_{\alpha_2}\mathcal{D}\|_{\mathcal{F}^{0,1}}). \end{aligned}$$

It suffices then to bound $\|\partial_{\alpha_i}\Omega\|_{\dot{\mathcal{F}}^{1,1}}$ and $\|\partial_{\alpha_i}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}$. From (4.24) we have that

$$\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq 2A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}} + A_\mu\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}, \tag{4.56}$$

$$\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} \leq 2\|BR_1\|_{\dot{\mathcal{F}}^{1,1}} + 2\|BR_3\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}} + 2\|BR_3\|_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}}.$$

Using an analogous bound to (4.48), it follows that

$$\begin{aligned} \|BR_1\|_{\dot{\mathcal{F}}^{1,1}} = \int_{\mathbb{R}^2}|\xi||\widehat{BR_1}(\xi)|d\xi \leq{}& \frac{1}{2}\sum_{n\geq 0}\int|\xi||\widehat{\omega_3}(\cdot)| * (*^{2n}|\cdot||\hat{f}(\cdot)|)(\xi)d\xi\\ &+\frac{1}{2}\sum_{n\geq 0}\int|\xi||\widehat{\omega_2}(\cdot)| * (*^{2n+1}|\cdot||\hat{f}(\cdot)|)(\xi)d\xi. \end{aligned}$$

By the product rule, we can distribute the multiplier $|\xi|$ to each term in the convolution to obtain

$$\|BR_1\|_{\dot{\mathcal{F}}^{1,1}} \leq \frac{1}{2}\sum_{n\geq 1} 2n \int |\widehat{\omega_3}(\cdot)| * (*^{2n-1}|\cdot||\hat{f}(\cdot)| * |\cdot|^2|\hat{f}(\cdot)|)(\xi)d\xi$$
$$+ \frac{1}{2}\sum_{n\geq 0}(2n+1) \int |\widehat{\omega_2}(\cdot)| * (*^{2n}|\cdot||\hat{f}(\cdot)| * |\cdot|^2|\hat{f}(\cdot)|)(\xi)d\xi$$
$$+ \frac{1}{2}\sum_{n\geq 0} \int (|\cdot||\widehat{\omega_3}(\cdot)|) * (*^{2n}|\cdot||\hat{f}(\cdot)|)(\xi)d\xi$$
$$+ \frac{1}{2}\sum_{n\geq 0} \int (|\cdot||\widehat{\omega_2}(\cdot)|) * (*^{2n+1}|\cdot||\hat{f}(\cdot)|)(\xi)d\xi.$$

Using Young's inequality, we finally obtain that

$$\|BR_1\|_{\dot{\mathcal{F}}^{1,1}} \leq \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{\left(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}\right)^2}\|\omega_3\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}} + \frac{1+\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}{2\left(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}\right)^2}\|\omega_2\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}}$$
$$+ \frac{1}{2}\frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}\|\omega_2\|_{\dot{\mathcal{F}}^{1,1}} + \frac{1}{2}\frac{1}{1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}\|\omega_3\|_{\dot{\mathcal{F}}^{1,1}}.$$

Proceeding in a similar way we have that

$$\|BR_3\|_{\dot{\mathcal{F}}^{1,1}} \leq \frac{1}{2}\frac{1}{1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}(\|\omega_1\|_{\dot{\mathcal{F}}^{1,1}} + \|\omega_2\|_{\dot{\mathcal{F}}^{1,1}})$$
$$+ \frac{\|f\|_{\dot{\mathcal{F}}^{1,1}}}{\left(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}\right)^2}\|f\|_{\dot{\mathcal{F}}^{2,1}}(\|\omega_1\|_{\mathcal{F}^{0,1}} + \|\omega_2\|_{\mathcal{F}^{0,1}}).$$

From the bounds in Proposition 4.4.1 we can write the above estimates as follows

$$\|BR_1\|_{\dot{\mathcal{F}}^{1,1}} \leq A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}}\frac{\left(1+\|f\|^2_{\dot{\mathcal{F}}^{1,1}}\right)\left(1+6A_\mu C_2\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)}{\left(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}\right)^2}$$
$$+ \frac{1}{2}S_1\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu\frac{1}{2}S_1\left(\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} + \|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}\right). \tag{4.57}$$

$$\|BR_3\|_{\dot{\mathcal{F}}^{1,1}} \leq 4C_1A_\rho S_1^2\|f\|_{\dot{\mathcal{F}}^{2,1}} + \frac{1}{2}\frac{1}{1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}}}(\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + \|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}}). \tag{4.58}$$

Then, using (4.57) and (4.58) as well as the estimates from Proposition 4.4.1, we obtain

$$\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} \leq 2\|BR_1\|_{\dot{\mathcal{F}}^{1,1}} + 2\|BR_3\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}} + 2\|BR_1\|_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}}$$
$$\leq S_1\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_1\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_1\|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}$$
$$+ 2C_3A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}} + 2S_1C_1A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}} + 8S_1^2C_1A_\rho\|f\|_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}}$$
$$+ S_1\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + S_1\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}}$$
$$\leq 2S_1\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + S_1\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_1\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_1\|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}$$
$$+ 2S_1A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)$$

Recalling the definition of S_1 and S_2 (4.49), from here we can write that

$$\begin{aligned}\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} \leq\ & 2S_2\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + S_2\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_2\|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}\\ &+ 2A_\rho S_2\|f\|_{\dot{\mathcal{F}}^{2,1}}\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right),\end{aligned}$$

and analogously,

$$\begin{aligned}\|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} \leq\ & 2S_2\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + S_2\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_2\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}}\\ &+ 2A_\rho S_2\|f\|_{\dot{\mathcal{F}}^{2,1}}\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right).\end{aligned}$$

We conclude that

$$\begin{aligned}\|\partial_{\alpha_1}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} \leq\ & S_2(2 + A_\mu S_2)\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + S_2(1 + 2A_\mu S_2)\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}}\\ &+ 2S_2^2A_\mu A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right),\\ \|\partial_{\alpha_2}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} \leq\ & S_2(2 + A_\mu S_2)\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + S_2(1 + 2A_\mu S_2)\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}}\\ &+ 2S_2^2A_\mu A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right).\end{aligned}$$

Now, we will introduce these inequalities into (4.56) to close the estimates. First, we have that

$$\begin{aligned}\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq\ & 2A_\rho\left(1 + S_2^2A_\mu^2\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)\right)\|f\|_{\dot{\mathcal{F}}^{2,1}}\\ &+ A_\mu S_2(2 + A_\mu S_2)\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu S_2(1 + 2A_\mu S_2)\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}},\end{aligned}$$

which implies that

$$\begin{aligned}\|\partial_{\alpha_1}\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq\ & \frac{A_\mu S_2(1 + 2A_\mu S_2)}{1 - A_\mu S_2(2 + A_\mu S_2)}\|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{1,1}}\\ &+ 2A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\frac{1 + S_2^2A_\mu^2\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)}{1 - A_\mu S_2(2 + A_\mu S_2)}.\end{aligned}$$

The above inequality combined with the analogous one for $\partial_{\alpha_2}\Omega$ yields that

$$\begin{aligned}\|\partial_{\alpha_i}\Omega\|_{\dot{\mathcal{F}}^{1,1}} &\leq 2A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\frac{1}{1 - \frac{A_\mu S_2(1+2A_\mu S_2)}{1-A_\mu S_2(2+A_\mu S_2)}}\frac{1 + S_2^2A_\mu^2\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)}{1 - A_\mu S_2(2 + A_\mu S_2)}\\ &= 2A_\rho\|f\|_{\dot{\mathcal{F}}^{2,1}}\frac{1 + S_2^2A_\mu^2\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)}{1 - 3A_\mu S_2(1 + A_\mu S_2)}.\end{aligned}$$

By denoting

$$C_4 = \frac{1 + S_2^2A_\mu^2\left(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}\right)}{1 - 3A_\mu S_2(1 + A_\mu S_2)},$$

we conclude that

$$\|\partial_{\alpha_i}\Omega\|_{\dot{\mathcal{F}}^{1,1}} \leq 2A_\rho C_4\|f\|_{\dot{\mathcal{F}}^{2,1}}, \tag{4.59}$$

and therefore

$$\begin{aligned}\|\partial_{\alpha_i}\mathcal{D}\|_{\dot{\mathcal{F}}^{1,1}} &\leq 2A_\rho S_2\|f\|_{\dot{\mathcal{F}}^{2,1}}\Big(3(1 + A_\mu S_2)C_4 + A_\mu S_2(C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}})\Big)\\ &= 2A_\rho C_5\|f\|_{\dot{\mathcal{F}}^{1,1}}\|f\|_{\dot{\mathcal{F}}^{2,1}},\end{aligned}$$

where we have denoted

$$C_5 = \frac{S_2}{\|f\|_{\dot{\mathcal{F}}^{1,1}}} \frac{3 + A_\mu S_2(3 + C_3 + C_1 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}})}{1 - 3A_\mu S_2(1 + A_\mu S_2)}.$$

Thus, the estimates for the vorticity are

$$\begin{aligned}
\|\omega_i\|_{\dot{\mathcal{F}}^{1,1}} &\leq 2A_\rho C_4 \|f\|_{\dot{\mathcal{F}}^{2,1}}, \quad i = 1, 2, \\
\|\omega_3\|_{\dot{\mathcal{F}}^{1,1}} &\leq 4A_\mu A_\rho \|f\|^2_{\dot{\mathcal{F}}^{1,1}} \|f\|_{\dot{\mathcal{F}}^{2,1}} (C_5 + 3C_2).
\end{aligned}$$

Finally, we estimate the quantity $\|\Omega\|_{\dot{\mathcal{F}}^{2,1}}$.

$$\begin{aligned}
\|\Omega\|_{\dot{\mathcal{F}}^{2,1}} &\leq A_\mu \|\partial_{\alpha_1}\mathcal{D}(\Omega)\|_{\dot{\mathcal{F}}^{1,1}} + A_\mu \|\partial_{\alpha_2}\mathcal{D}(\Omega)\|_{\dot{\mathcal{F}}^{1,1}} + 2A_\rho \|f\|_{\dot{\mathcal{F}}^{2,1}} \\
&\leq 6A_\mu S_2(1 + A_\mu S_2) + 2A_\rho(1 + 2S_2^2 A_\mu(C_1 + C_3 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}}))\|f\|_{\dot{\mathcal{F}}^{2,1}}.
\end{aligned}$$

Therefore,

$$\|\Omega\|_{\dot{\mathcal{F}}^{2,1}} \leq 2A_\rho \frac{1 + 2S_2^2 A_\mu(C_1 + C_3 + 4S_1C_1\|f\|_{\dot{\mathcal{F}}^{1,1}})}{1 - 6A_\mu S_2(1 + A_\mu S_2)} \|f\|_{\dot{\mathcal{F}}^{2,1}}.$$

This concludes the proof. □

Remark 4.4.3. *Because we actually have the triangle inequality*

$$|\xi|^s \leq |\xi - \xi_1|^s + \sum_{k=1}^{m} |\xi_j - \xi_{j+1}|^s + |\xi_{m+1}|^s$$

for all $0 < s \leq 1$, *notice that the same arguments as above can be used to show that*

$$\|\omega_1\|_{\dot{\mathcal{F}}^{s,1}_\nu} = \|\partial_{\alpha_2}\Omega\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq 2A_\rho C_{4,\nu} \|f\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}$$

and

$$\|\omega_3\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq 4A_\mu A_\rho \|f\|^2_{\dot{\mathcal{F}}^{1,1}} \|f\|_{\dot{\mathcal{F}}^{2,1}} (C_{5,\nu} + 3C_{2,\nu})$$

where the constants $C_{2,\nu}$, $C_{4,\nu}$ *and* $C_{5,\nu}$ *now depend on* $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ *rather than* $\|f\|_{\dot{\mathcal{F}}^{1,1}}$.

4.5 Instant Analyticity of f

We dedicate this section to proving the norm decrease inequality (4.63) which will be needed to obtain the global existence results. Note that (4.63) states that the interface function becomes instantly analytic given medium-sized initial data $f_0 \in \dot{\mathcal{F}}^{1,1}$. Precisely, we show the following:

Proposition 4.5.1. *Assume the initial data* f_0 *satisfies that*

$$\sigma\left(\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\right) > 0, \tag{4.60}$$

where

$$\sigma\left(\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\right) = -\nu + A_\rho \left(1 - 2\Big(\frac{2B_1 + B_2 - B_2\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}}{(1-\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}})^2}\Big)\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}} \right.$$
$$\left. -A_\mu\Big(\frac{12C_2 + 2C_5 - 2C_5\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}}{(1-\|f_0\|^2_{\dot{\mathcal{F}}^{1,1}})^2}\Big)\|f_0\|^3_{\dot{\mathcal{F}}^{1,1}} - 2A_\mu C_5\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\right).$$

All the constants above are defined precisely in (4.38), (4.43), (4.52), *and* (4.55), *which are given during the proofs of the previous estimates. Then*

$$\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \sigma\left(\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\right)\int_0^t \|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)d\tau \le \|f_0\|_{\dot{\mathcal{F}}^{1,1}}.$$

Proof. We will use the evolution equation (4.19) and (4.20). Differentiating the quantity $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) &= \frac{d}{dt}\left(\int |\xi| e^{\nu t|\xi|}|\hat{f}(\xi)|d\xi\right) \\
&\le \nu\int |\xi|^2 e^{t\nu|\xi|}|\hat{f}(\xi)|d\xi + \int |\xi|e^{\nu t|\xi|}\frac{1}{2}\Big(\frac{\hat{f}_t\overline{\hat{f}} + \hat{f}\overline{\hat{f}_t}}{|\hat{f}(\xi)|}\Big)d\xi \\
&\le (\nu - A_\rho)\int |\xi|^2 e^{\nu t|\xi|}|\hat{f}(\xi)|d\xi + \int |\xi|e^{\nu t|\xi|}|\widehat{N(f)}(\xi)|d\xi.
\end{aligned}$$

Hence, using the decomposition (4.20), we can use the Fourier arguments as earlier, such as (4.48), to pointwise bound the nonlinear term

$$|\widehat{N(f)}(\xi)| \le |\widehat{N_1(f)}(\xi)| + |\widehat{N_2(f)}(\xi)| + |\widehat{N_3(f)}(\xi)|$$

in frequency space. The latter two terms are bounded by

$$|\widehat{N_2(f)}(\xi)| \le \sum_{n\ge 1}\Big((*^{2n}|\cdot||\hat{f}(\cdot)|) * |\cdot||\hat{\Omega}(\xi)|\Big)(\xi) \tag{4.61}$$

and

$$|\widehat{N_3(f)}(\xi)| \le \frac{A_\mu}{2}\sum_{n\ge 1}\Big((*^{2n}|\cdot||\hat{f}(\cdot)|) * |\cdot||\widehat{\mathcal{D}(\Omega)}(\xi)|\Big)(\xi). \tag{4.62}$$

The estimate on $\widehat{N_1(f)}(\xi)$ is done in Section 4.4:

$$\|N_1\|_{\dot{\mathcal{F}}^{1,1}_\nu} = \frac{A_\mu}{2}\|\mathcal{D}(\Omega)\|_{\dot{\mathcal{F}}^{2,1}_\nu} \le 2A_\rho A_\mu C_5\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}.$$

For the other two nonlinear terms, using the triangle inequality

$$|\xi| \le |\xi - \xi_1| + |\xi_1 - \xi_2| + \cdots + |\xi_{2n}|,$$

we obtain that

$$e^{\nu t|\xi|}\leq e^{\nu t|\xi-\xi_1|}e^{\nu t|\xi_1-\xi_2|}\dots e^{\nu t|\xi_{2n}|}$$

and therefore

$$\begin{aligned}\int|\xi|e^{\nu t|\xi|}|\widehat{N_2(f)}(\xi)|d\xi &\leq \sum_{n\geq 1}\int|\xi|e^{\nu t|\xi|}\Big((*^{2n}|\cdot||\hat{f}(\cdot)|)*|\cdot||\hat{\Omega}(\xi)|\Big)(\xi)d\xi\\ &\leq \sum_{n\geq1}2n\int\Big((*^{2n-1}|\cdot|e^{\nu t|\cdot|}|\hat{f}(\cdot)|)*|\cdot|e^{\nu t|\cdot|}|\hat{\Omega}(\xi)|*|\cdot|^2e^{\nu t|\cdot|}|\hat{f}(\xi)|\Big)(\xi)d\xi\\ &\quad+\sum_{n\geq1}\int\Big((*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat{f}(\cdot)|)*|\cdot|^2e^{\nu t|\cdot|}|\hat{\Omega}(\xi)|\Big)(\xi)d\xi\\ &\leq \sum_{n\geq1}2n\|f\|^{2n-1}_{\dot{\mathcal{F}}^{1,1}_\nu}\|\Omega\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}+\sum_{n\geq1}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu}\|\Omega\|_{\dot{\mathcal{F}}^{2,1}_\nu}\\ &\leq 2A_\rho B_1\sum_{n\geq1}2n\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}+2A_\rho B_2\sum_{n\geq1}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}\end{aligned}$$

Similarly,

$$\begin{aligned}&\int|\xi|e^{\nu t|\xi|}|\widehat{N_3(f)}(\xi)|d\xi\\ &\qquad\leq\frac{A_\mu}{2}\Big(\sum_{n\geq1}2n\|f\|^{2n-1}_{\dot{\mathcal{F}}^{1,1}_\nu}\|\mathcal{D}(\Omega)\|_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}+\sum_{n\geq1}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu}\|\mathcal{D}(\Omega)\|_{\dot{\mathcal{F}}^{2,1}_\nu}\Big)\end{aligned}$$

for the N_3 nonlinear term. Plugging in the estimates (4.41) and (4.54) for $\mathcal{D}(\Omega)$, we obtain

$$\int|\xi|e^{\nu t|\xi|}|\widehat{N_3(f)}(\xi)|d\xi\leq A_\mu A_\rho\Big(12C_2\sum_{n\geq1}n\|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}+2C_5\sum_{n\geq1}\|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu}\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}\Big).$$

By collecting the previous estimates, we obtain that

$$\frac{d}{dt}\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t)\leq-\sigma\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}, \tag{4.63}$$

where

$$\begin{aligned}\sigma=-\nu+A_\rho-2A_\rho A_\mu C_5\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}-2A_\rho B_1\sum_{n\geq1}2n\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu}-2A_\rho B_2\sum_{n\geq1}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu}\\ -A_\mu A_\rho\Big(12C_2\sum_{n\geq1}n\|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu}+2C_5\sum_{n\geq1}\|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu}\Big).\end{aligned} \tag{4.64}$$

Writing the sums in a definite form,

$$\begin{aligned}\sigma=-\nu+A_\rho-2A_\rho A_\mu C_5\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}-2A_\rho\Big(\frac{2B_1+B_2-B_2\|f\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}_\nu})^2}\Big)\|f\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}\\ -A_\mu A_\rho\Big(\frac{12C_2+2C_5-2C_5\|f\|^2_{\dot{\mathcal{F}}^{1,1}_\nu}}{(1-\|f\|^2_{\dot{\mathcal{F}}^{1,1}_\nu})^2}\Big)\|f\|^3_{\dot{\mathcal{F}}^{1,1}_\nu}.\end{aligned} \tag{4.65}$$

This completes the proof. □

Remark 4.5.2. *We would also now like to comment on our estimate in the case of no viscosity jump, which is the regime considered in [30]. Setting* $A_\mu = 0$*, we obtain from* (4.64) *that*

$$\sigma = A_\rho \left(1 - 2\sum_{n\geq 1}(2n+1)\|f_0\|^{2n}_{\dot{\mathcal{F}}^{1,1}} \right).$$

Hence, σ *is a positive constant for* $\|f_0\|_{\dot{\mathcal{F}}^{1,1}}$ *satisfying*

$$2\sum_{n\geq 1}(2n+1)\|f_0\|^{2n}_{\dot{\mathcal{F}}^{1,1}} < 1.$$

This is the condition for the 2D case in [30]. However, here we show that this condition is also sufficient in the 3D case, thereby improving the previous results.

4.6 L^2 maximum principle

For completeness, we present the proof of a L^2 maximum principle for Muskat solutions in the viscosity jump regime. Given that the viscosities and densities of both fluids are constant on each domain (4.3), from Darcy's law (4.1) one obtains that the flow is irrotational away from the free boundary:

$$\text{curl}\, u(x,t) = 0, \qquad x \in D^1(t) \cup D^2(t).$$

Thus we find that the velocity comes from a potential ϕ

$$u = \nabla\phi, \tag{4.66}$$

and since the flow is incompressible we obtain that

$$\Delta\phi = 0.$$

Now, integration by parts shows that

$$0 = \mu^i \int_{D^i} \phi\Delta\phi\, dx = -\mu^i \int_{D^i} \nabla\phi\cdot\nabla\phi\, dx + \mu^i \int_{\partial D^i} \nabla\phi\cdot n\phi\, d\sigma,$$

so using (4.66) it reads as

$$-\mu^i \int_{D^i} |u|^2\, dx + \int_{\partial D^i} u\cdot n\mu^i\phi\, d\sigma = 0.$$

Recalling that the normal velocity is continuous across the boundary due to the incompressibility condition, by adding the balance of both domains we can write

$$-\int_{\mathbb{R}^3} \mu|u|^2\, dx + \int_{\partial D} u\cdot n(\mu^2\phi^2 - \mu^1\phi^1)\, d\sigma = 0. \tag{4.67}$$

Here ϕ^i is the potential in D^i. Introducing (4.66) in (4.1) we find that

$$\mu^i \phi^i = -p - \rho^i x_3.$$

From this and the continuity of the pressure along the boundary we obtain that

$$-\int_{\mathbb{R}^3} \mu |u|^2 \, dx = (\rho_2 - \rho_1) \int_{\partial D} u \cdot n x_3 \, d\sigma. \tag{4.68}$$

If the boundary is described as a graph

$$\partial D(t) = \{(\alpha, f(\alpha, t)) \in \mathbb{R}^3 : \alpha \in \mathbb{R}^3\},$$

since it moves with the flow one has that

$$\begin{aligned} f_t(\alpha) &= u(\alpha, f(\alpha)) \cdot (-\partial_{\alpha_1} f(\alpha), -\partial_{\alpha_2} f(\alpha), 1)) \\ &= u(\alpha, f(\alpha)) \cdot n(\alpha) \sqrt{1 + (\partial_{\alpha_1} f(\alpha))^2 + (\partial_{\alpha_2} f(\alpha))^2}. \end{aligned}$$

Going back to (4.68) we find that

$$(\rho_2 - \rho_1) \int_{\mathbb{R}^2} f_t(\alpha) f(\alpha) \, d\alpha + \int_{\mathbb{R}^3} \mu |u|^2 \, dx = 0,$$

so by integration in time we finally obtain the L^2 maximum principle

$$(\rho^2 - \rho^1) \|f\|_{L^2}^2(t) + 2 \int_{\mathbb{R}^3} \mu |u|^2 \, dx = (\rho_2 - \rho_1) \|f_0\|_{L^2}^2.$$

4.7 Uniqueness

Proposition 4.7.1. *Consider two solutions f and g to the Muskat problem with initial data $f_0, g_0 \in L^2 \cap \dot{\mathcal{F}}^{1,1}$ that satisfy the condition* (4.60). *Then,*

$$\frac{d}{dt} \|f - g\|_{\mathcal{F}^{0,1}} \leq -C \|f - g\|_{\dot{\mathcal{F}}^{1,1}},$$

and moreover, $\|f - g\|_{L^\infty} = 0$.

Proof. Using (4.19), we can write, as before

$$\begin{aligned} \frac{d}{dt} \|f - g\|_{\mathcal{F}^{0,1}_\nu} &= \int_{\mathbb{R}^2} \frac{1}{2} \frac{\overline{\widehat{(f-g)}}(\xi) \partial_t \widehat{(f-g)}(\xi) + \widehat{(f-g)}(\xi) \partial_t \overline{\widehat{(f-g)}}(\xi)}{|\widehat{(f-g)}(\xi)|} \\ &\leq (\nu - A_\rho) \|f - g\|_{\dot{\mathcal{F}}^{1,1}} + \sum_{i=1}^{3} \int |\widehat{N_i(f-g)}(\xi)| d\xi \end{aligned}$$

where N_i are the nonlinear terms given by (4.20). For example,

$$\int |\widehat{N_1(f-g)}(\xi)| d\xi = \frac{A_\mu}{2} \int_{\mathbb{R}^2} |\Lambda \widehat{D(\Omega(f))}(\xi) - \Lambda \widehat{D(\Omega(g))}(\xi)| d\xi,$$

where $\Omega(f)$ is the term Ω in the case of the solution f and similarly for $\Omega(g)$. We define the terms N_2 and N_3 later. As previously done, we use the decomposition (4.16), where $\partial_{\alpha_i}\mathcal{D}(\Omega)$ is given by (4.23). Hence, we can write for $i = 1$

$$\int_{\mathbb{R}^2} |\partial_{\alpha_1}\widehat{D(\Omega(f))}(\xi) - \partial_{\alpha_1}\widehat{D(\Omega(g))}(\xi)|d\xi$$
$$\leq \int_{\mathbb{R}^2} |\widehat{BR_1(f)}(\xi) - \widehat{BR_1(g)}(\xi)|d\xi + \int_{\mathbb{R}^2} |\widehat{BR_3(f)\partial_{\alpha_1}f}(\xi) - \widehat{BR_3(g)\partial_{\alpha_1}g}(\xi)|d\xi.$$

First, we consider the $BR_1 = BR_{11} + BR_{12}$ term. Using the Taylor expansion, we can write $BR_{11}(f)$ as

$$\begin{aligned} BR_{11}(f) &= -\frac{1}{8\pi}\sum_{n\geq 0}\text{p.v.}\int_{\mathbb{R}^2}\Big(\omega_3(f)(\alpha-\beta)(-1)^n a_n(\Delta_\beta f(\alpha))^{2n} \\ &\quad - \omega_3(f)(\alpha+\beta)(-1)^n a_n(\Delta_{-\beta} f(\alpha))^{2n}\Big)\frac{\beta_2 d\beta}{|\beta|^3} \\ &\stackrel{\text{def}}{=} BR_{11}^{+}(f) - BR_{11}^{-}(f). \end{aligned}$$

Next, we get that the integrand of the n-th term in $BR_{11}(f) - BR_{11}(g)$ is given by

$$\begin{aligned} &BR_{11}^{+}(f) - BR_{11}^{+}(g) = (-1)^n a_n(p_1 p_2^{2n} - q_1 q_2^{2n}) \\ &= (-1)^n a_n(p_1 p_2^{2n} - q_1 p_2^{2n} + q_1 p_2^{2n} - q_1 q_2 p_2^{2n-1} + q_1 q_2 p_2^{2n-1} - \ldots + q_1 q_2^{2n-1} p_2 - q_1 q_2^{2n}) \\ &\qquad = (-1)^n a_n\Big((p_1 - q_1)p_2^{2n} + q_1 p_2^{2n-1}(p_2 - q_2) + q_1 q_2 p_2^{2n-2}(p_2 - q_2) + \\ &\qquad\qquad \ldots + q_1 q_2^{2n-1}(p_2 - q_2)\Big) \end{aligned} \tag{4.69}$$

where $p_1 = \omega_3(f)(\alpha-\beta)$, $p_2 = \Delta_\beta(f)(\alpha)$, $q_1 = \omega_3(g)(\alpha-\beta)$ and $q_2 = \Delta_\beta(g)(\alpha)$. We do the same for BR_{11}^{-} by defining $p_{-1} = \omega_3(f)(\alpha+\beta)$, $p_{-2} = \Delta_{-\beta}(f)(\alpha)$, $q_{-1} = \omega_3(g)(\alpha+\beta)$ and $q_{-2} = \Delta_{-\beta}(g)(\alpha)$. Next, using the Fourier arguments to bound BR_{11} as in Section 4.4, we can obtain that

$$\begin{aligned} &|\widehat{BR_{11}(f)}(\xi) - \widehat{BR_{11}(g)}(\xi)| \\ &\quad \leq \frac{1}{2}\sum_{n\geq 0} |\widehat{\omega_3(f) - \omega_3(g)}| * (*^{2n}|\cdot||\hat{f}|) + |\widehat{\omega_3(g)}| * (*^{2n-1}|\cdot||\hat{f}|) * |\cdot||\widehat{f-g}(\cdot)| \\ &\qquad + \ldots + |\widehat{\omega_3(g)}| * (*^{2n-1}|\cdot||\hat{g}|) * |\cdot||\widehat{f-g}(\cdot)|. \end{aligned} \tag{4.70}$$

Hence, applying Young's inequality,

$$\begin{aligned} &\int_{\mathbb{R}^2} |\widehat{BR_{11}(f)}(\xi) - \widehat{BR_{11}(g)}(\xi)|d\xi \\ &\quad \leq \frac{1}{2}\sum_{n\geq 0} \|\omega_3(f) - \omega_3(g)\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}}^{2n} + \|\omega_3(g)\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}}^{2n-1}\|f-g\|_{\dot{\mathcal{F}}^{1,1}} \\ &\qquad + \ldots + \|\omega_3(g)\|_{\mathcal{F}^{0,1}}\|g\|_{\dot{\mathcal{F}}^{1,1}}^{2n-1}\|f-g\|_{\dot{\mathcal{F}}^{1,1}}. \end{aligned} \tag{4.71}$$

Next,

$$
\begin{aligned}
&\omega_3(f) - \omega_3(g) \\
&\quad = \partial_{\alpha_2} D(\Omega(f))\partial_{\alpha_1} f - \partial_{\alpha_2} D(\Omega(g))\partial_{\alpha_1} g - \partial_{\alpha_1} D(\Omega(f))\partial_{\alpha_2} f + \partial_{\alpha_1} D(\Omega(g))\partial_{\alpha_2} g \\
&\qquad = \partial_{\alpha_2}(D(\Omega(f)) - D(\Omega(g)))\partial_{\alpha_1} f + \partial_{\alpha_2} D(\Omega(g))(\partial_{\alpha_1}(f-g)) \\
&\qquad\qquad - \partial_{\alpha_1}(D(\Omega(f)) - D(\Omega(g)))\partial_{\alpha_2} f - \partial_{\alpha_1} D(\Omega(g))(\partial_{\alpha_2}(f-g)).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\omega_3(f) - \omega_3(g)\|_{\mathcal{F}^{0,1}} \\
&\quad \le \|\partial_{\alpha_1}(D(\Omega(f)) - D(\Omega(g)))\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}} + \|\partial_{\alpha_2}(D(\Omega(f) - \Omega(g)))\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}} \\
&\qquad + \|\partial_{\alpha_1} D(\Omega(g))\|_{\mathcal{F}^{0,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}} + \|\partial_{\alpha_2} D(\Omega(g))\|_{\mathcal{F}^{0,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned} \tag{4.72}
$$

Furthermore, for BR_{12} we similarly obtain

$$
\begin{aligned}
&\|BR_{12}(f) - BR_{12}\|_{\mathcal{F}^{0,1}} \\
&\quad \le \frac{1}{2}\sum_{n\ge 0} \|\partial_{\alpha_1}\Omega(g) - \partial_{\alpha_1}\Omega(g)\|_{\dot{\mathcal{F}}^{0,1}}\|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}} + \|\partial_{\alpha_1}\Omega(g)\|_{\mathcal{F}^{0,1}}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}} \\
&\qquad + \ldots + \|\partial_{\alpha_1}\Omega(g)\|_{\mathcal{F}^{0,1}}\|g\|^{2n}_{\dot{\mathcal{F}}^{1,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}
$$

Next, for the BR_3 integral term

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\widehat{BR_3(f)\partial_{\alpha_1}} f(\xi) - \widehat{BR_3(g)\partial_{\alpha_1}} g(\xi)|d\xi \\
&\qquad \le \|BR_3(f) - BR_3(g)\|_{\mathcal{F}^{0,1}}\|f\|_{\dot{\mathcal{F}}^{1,1}} + \|BR_3(g)\|_{\dot{\mathcal{F}}^{0,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\|BR_3(f) - BR_3(g)\|_{\mathcal{F}^{0,1}} \\
&\quad \le \frac{1}{2}\sum_{i=1,2}\sum_{n\ge 0} \|\partial_{\alpha_i}\Omega(f) - \partial_{\alpha_i}\Omega(g)\|_{\mathcal{F}^{0,1}}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}} + \|\partial_{\alpha_i}\Omega(g)\|_{\mathcal{F}^{0,1}}\|f\|^{2n-1}_{\dot{\mathcal{F}}^{1,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}} \\
&\qquad + \ldots + \|\partial_{\alpha_i}\Omega(g)\|_{\mathcal{F}^{0,1}}\|g\|^{2n-1}_{\dot{\mathcal{F}}^{1,1}}\|f-g\|_{\dot{\mathcal{F}}^{1,1}}.
\end{aligned}
$$

The key point to note here is that these estimates are precisely those used to prove the vorticity estimates in the norm $\dot{\mathcal{F}}^{1,1}$ in Proposition 4.4.2 if we replace the quantities

$$
\begin{aligned}
\|\partial_{\alpha_i}\Omega(f)\|_{\dot{\mathcal{F}}^{1,1}} \text{ or } \|\partial_{\alpha_i}\Omega(g)\|_{\dot{\mathcal{F}}^{1,1}} &\leftrightarrow \|\partial_{\alpha_i}\Omega(g) - \partial_{\alpha_i}\Omega(g)\|_{\mathcal{F}^{0,1}} \\
\|f\|_{\dot{\mathcal{F}}^{2,1}} \text{ or } \|g\|_{\dot{\mathcal{F}}^{2,1}} &\leftrightarrow \|f-g\|_{\dot{\mathcal{F}}^{1,1}},
\end{aligned} \tag{4.73}
$$

and notice by counting terms that the computation of (4.69) creates the same effect on the estimates as the effect created by the triangle inequality (or product rule) in the case of estimates of Proposition 4.4.2. Therefore, continuing to compute the estimates for uniqueness as above and comparing with the estimates of Section 4.4 and 4.5 by using the substitutions (4.73), we obtain the analogous estimate, for example:

$$\|\partial_{\alpha_i}\Omega(f)-\partial_{\alpha_i}\Omega(g)\|_{\mathcal{F}^{0,1}}\leq 2A_\rho C_4\|f-g\|_{\dot{\mathcal{F}}^{1,1}}.$$

These vorticity estimates and performing similar computations on the nonlinear terms N_i, we can see that

$$\frac{d}{dt}\|f-g\|_{\mathcal{F}^{0,1}}\leq -\sigma\|f-g\|_{\dot{\mathcal{F}}^{1,1}}$$

where σ is the same positive constant as in Proposition 4.5.1. It can be seen by the swap of terms described above in (4.73). □

4.8 Regularization

We describe now the regularization of the system together with the limit process to get bona-fide and not just a priori estimates for the Muskat problem. We denote the heat kernel ζ_ε as an approximation to the identity where ε plays the role of time in such a way that ζ_ε converges to the identity as $\varepsilon\to 0^+$. We consider the following regularization of the system

$$\partial_t f^\varepsilon=-\frac{A_\rho}{2}\Lambda(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)+\zeta_\varepsilon*(N(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon,\Omega^\varepsilon)),\quad f^\varepsilon(x,0)=(\zeta_\varepsilon*f_0)(x). \tag{4.74}$$

where $N(\cdot,\cdot)$ is given by (4.19) and (4.20), and Ω^ε by

$$\Omega^\varepsilon(\alpha,t)=A_\mu\mathcal{D}^\varepsilon(\Omega^\varepsilon)(\alpha,t)-2A_\rho\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon(\alpha,t). \tag{4.75}$$

The operator $\mathcal{D}^\varepsilon(\Omega^\varepsilon)$ is written as follows

$$\mathcal{D}^\varepsilon(\Omega^\varepsilon)(\alpha)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\frac{\beta}{|\beta|}\cdot\nabla_\alpha(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha-\beta)-\Delta_\beta(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha)}{(1+(\Delta_\beta(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha))^2)^{3/2}}\frac{\Omega^\varepsilon(\alpha-\beta)}{|\beta|^2}d\beta. \tag{4.76}$$

Integration by parts also provides the identities

$$\begin{aligned}\partial_{\alpha_i}\mathcal{D}^\varepsilon(\Omega^\varepsilon)=&-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\Delta_\beta(\partial_{\alpha_i}(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon))(\alpha)}{(1+(\Delta_\beta(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha))^2)^{3/2}}\frac{\beta\cdot\nabla_\alpha\Omega^\varepsilon(\alpha-\beta)}{|\beta|^2}d\beta\\&+\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\frac{\beta}{|\beta|}\cdot\nabla_\alpha(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha-\beta)-\Delta_\beta(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha)}{(1+(\Delta_\beta(\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon)(\alpha))^2)^{3/2}}\frac{\partial_{\alpha_i}\Omega^\varepsilon(\alpha-\beta)}{|\beta|^2}d\beta.\end{aligned} \tag{4.77}$$

Then it is easy to estimate Ω^ϵ as in Section 4.4 in terms of $\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon$ with the condition $\|\zeta_\varepsilon*\zeta_\varepsilon*f^\varepsilon\|_{\dot{\mathcal{F}}^{1,1}}(t)<1$. These estimates provide a local existence result using the classical Picard theorem on the Banach space $C([0,T_\varepsilon];H^4)$. We find the abstract evolution system given by $\partial_t f^\varepsilon=G(f^\varepsilon)$ where G is Lipschitz on the open set $\{g(x)\in H^4:\|g\|_{\dot{\mathcal{F}}^{1,1}}<1\}$. We remember that $f^\varepsilon(x,0)\in H^4$ due to $f_0\in L^2$. The next step is to reproduce estimate (4.63) for $s=1$. As the convolutions are taken with the heat kernel, it is easy to prove analyticity for f^ε so that for ν small enough we find that $\|f^\epsilon\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ bounded. Even more, we know that $\|f^\epsilon\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t)<k(|A_\mu|)$, as continuity in time provides that this quantity is close in

size to $\|f^\epsilon\|_{\dot{\mathcal{F}}^{1,1}_\nu}(0) = \|f^\epsilon\|_{\dot{\mathcal{F}}^{1,1}}(0) \le \|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k(|A_\mu|)$ if $T_\varepsilon > 0$ is small enough. Therefore, in checking its evolution as in Section 4.5 we find that

$$\frac{d}{dt}\|f^\epsilon\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) \le -C\|\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon\|_{\dot{\mathcal{F}}^{2,1}_\nu},$$

so that integration in time provides

$$\|f^\varepsilon\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + C\int_0^t \|\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)d\tau \le \|f_0\|_{\dot{\mathcal{F}}^{1,1}}. \tag{4.78}$$

Next we repeat the computations in Section 4.9 for the regularized system. It is possible to find that

$$\|f^\varepsilon\|_{L^2_\nu}(t) \le \|f_0\|_{L^2} \exp\Big(R(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\Big).$$

Energy estimates provide

$$\frac{d}{dt}\|f^\varepsilon\|^2_{H^4} \le P(\|\zeta_\varepsilon * f^\varepsilon\|^2_{H^4})$$

where P is a polynomial function. Then, using that

$$\|\zeta_\varepsilon * f^\varepsilon\|_{H^4} \le C(\varepsilon)\|f^\varepsilon\|_{L^2} \le C(\varepsilon)\|f_0\|_{L^2} \exp\Big(C(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\Big)$$

we are able to extend the solutions in $C([0,T];H^4)$ for any $T > 0$.

Next, we find a candidate for a solution by taking the limit $\varepsilon \to 0^+$ after proving that f^ϵ is Cauchy $L^\infty(0,T;\mathcal{F}^{0,1})$. From now on, we consider $\varepsilon \ge \varepsilon' > 0$. Then, as in Section 4.7, we are able to find that

$$\|f^\varepsilon - f^{\varepsilon'}\|_{\mathcal{F}^{0,1}}(t) \le \|\zeta_\varepsilon * f_0 - \zeta_{\varepsilon'} * f_0\|_{\mathcal{F}^{0,1}} + I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_0^t \frac{A_\rho}{2}\|\Lambda(\zeta_\varepsilon * \zeta_\varepsilon * f^{\varepsilon'} - \zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'})\|_{\mathcal{F}^{0,1}}(\tau)d\tau 87,$$

and

$$I_2(t) = \int_0^t \|\zeta_\varepsilon * N(\zeta_\varepsilon * \zeta_\varepsilon * f^{\varepsilon'}, \Omega^{\varepsilon'}) - \zeta_{\varepsilon'} * N(\zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}, \Omega^{\varepsilon'})\|_{\mathcal{F}^{0,1}}(s)ds.$$

As before, in order to get the inequality above, we use the decay from the dissipation term to absorb the bounds for $\zeta_\varepsilon * N(\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon, \Omega^\varepsilon) - \zeta_\varepsilon * N(\zeta_\varepsilon * \zeta_\varepsilon * f^{\varepsilon'}, \Omega^{\varepsilon'})$. Then, using the mean value theorem in the heat kernel on the Fourier side, it is possible to get

$$\|\zeta_\varepsilon * f_0 - \zeta_{\varepsilon'} * f_0\|_{\mathcal{F}^{0,1}} \le C\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\varepsilon^{1/2}. \tag{4.79}$$

Similarly

$$I_1(t) \le C\int_0^t \|\zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}^{2,1}}(s)ds\,\varepsilon^{1/2} \le C\|f_0\|_{\dot{\mathcal{F}}^{1,1}}\varepsilon^{1/2}.$$

A further splitting in the mollifiers, together with the inequality

$$\|\zeta_\varepsilon * \zeta_\varepsilon * f^{\varepsilon'}\|_{\mathcal{F}^{s,1}}(s) \leq \|\zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}^{s,1}}(s), \quad s \geq 0,$$

allows us to find for the nonlinear term, as before, the following bound

$$I_2(t) \leq C(\|f_0\|_{\dot{\mathcal{F}}^{1,1}}) \int_0^t \|\zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}^{2,1}}(s) ds\, \varepsilon^{1/2} \leq C(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\varepsilon^{1/2}.$$

It yields finally

$$\|f^\varepsilon - f^{\varepsilon'}\|_{\mathcal{F}^{0,1}}(t) \leq C(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\varepsilon^{1/2}, \tag{4.80}$$

so that we are done finding a limit $f \in L^\infty(0,T;\mathcal{F}^{0,1})$. The interpolation inequality

$$\|g\|^2_{\dot{\mathcal{F}}^{1,1}} \leq \|g\|_{\mathcal{F}^{0,1}}\|g\|_{\dot{\mathcal{F}}^{2,1}}$$

provides

$$\begin{aligned}\int_0^t \|\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon - \zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|^2_{\dot{\mathcal{F}}^{1,1}}(s)ds \leq \\ \int_0^t A(s)(\|\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon\|_{\dot{\mathcal{F}}^{2,1}}(s) + \|\zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|_{\dot{\mathcal{F}}^{2,1}}(s))ds,\end{aligned} \tag{4.81}$$

where

$$A(s) = \|\zeta_\varepsilon * \zeta_\varepsilon * (f^\varepsilon - f^{\varepsilon'})\|_{\mathcal{F}^{0,1}}(s) + \|\zeta_\varepsilon * \zeta_\varepsilon * f^{\varepsilon'} - \zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|_{\mathcal{F}^{0,1}}(s).$$

The first term in $A(s)$ is controlled by (4.80) and for the second term we apply a similar approach as in (4.79) to find

$$A(s) \leq C(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\varepsilon^{1/2}.$$

Using (4.78) in (4.81) we find finally

$$\int_0^t \|\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon - \zeta_{\varepsilon'} * \zeta_{\varepsilon'} * f^{\varepsilon'}\|^2_{\dot{\mathcal{F}}^{1,1}}(s)ds \leq C(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\varepsilon^{1/2}, \tag{4.82}$$

which provides strong convergence of $\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon$ to f in $L^2(0,T;\dot{\mathcal{F}}^{1,1})$.

Next we can extract a subsequence f^{ε_n} in such a way that

$$(\widehat{f^{\varepsilon_n}}(\xi,t), \exp(-8\pi^2\varepsilon_n|\xi|^2)\widehat{f^{\varepsilon_n}}(\xi,t)) \to (\widehat{f}(\xi,t), \widehat{f}(\xi,t))$$

pointwise for almost every $(\xi,t) \in \mathbb{R}^2 \times [0,T]$. Therefore for $t \in [0,T] \smallsetminus Z$ with measure $|Z| = 0$ it is possible to find the same pointwise for almost every $\xi \in \mathbb{R}^2$. Fatou's lemma allows us to conclude that for $t \in [0,T] \smallsetminus Z$ and

$$M(t) = \|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + C\int_0^t \|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}(s)$$

it is possible to obtain

$$M(t) \leq \liminf_{n\to\infty}\Big(\|f^{\varepsilon_n}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + C\int_0^t \|\zeta_{\varepsilon_n} * \zeta_{\varepsilon_n} * f^{\varepsilon_n}\|_{\dot{\mathcal{F}}^{2,1}_\nu}(s)ds\Big) \leq \|f_0\|_{\dot{\mathcal{F}}^{1,1}},$$

The strong convergence of $\zeta_\varepsilon * \zeta_\varepsilon * f^\varepsilon$ to f in $L^2(0,T;\dot{\mathcal{F}}^{1,1})$ together with the regularity found for f allow us to take the limit in equations (4.74,4.75,4.76) to find f as a solution to the original Muskat equations (4.19-4.22). Now we use the approach in Section 4.6 to get the L^2 maximum principle for f.

4.9 Gain of L^2 Derivatives with Analytic Weight

In this section, we first show gain of L^2 regularity. In particular, we prove uniform bounds in $L^2_\nu = \mathcal{F}^{0,2}_\nu$, which will be used to show decay of analytic L^2 norms, and more prominently, the ill-posedness argument of Section 4.11:

Theorem 4.9.1. *Suppose $f_0 \in L^2 \cap \dot{\mathcal{F}}^{1,1}$ and $\|f_0\|_{\dot{\mathcal{F}}^{1,1}}$ satisfying the condition* (4.60)*. Then, $f(t) \in L^2_\nu$ instantly for all $t > 0$. Moreover*

$$\|f\|^2_{L^2_\nu}(t) \leq \|f_0\|^2_{L^2} \exp(R(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})),$$

with R a rational function. In particular, this implies that $f(t) \in H^s$ instantly for all $t > 0$.

Proof. Differentiating the

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_\nu}(t) = (\nu - A_\rho)\|f\|^2_{\dot{H}^{1/2}_\nu} + \frac{A_\mu}{2}\int |\xi| e^{2\nu t|\xi|}|\widehat{\mathcal{D}(\Omega)}(\xi)||\hat{f}(\xi)|d\xi + \int e^{2\nu t|\xi|}|\widehat{N_2}(\xi)||\hat{f}(\xi)|d\xi + \int e^{2\nu t|\xi|}|\widehat{N_3}(\xi)||\hat{f}(\xi)|d\xi.$$

We now bound the nonlinear terms. For example,

$$\int |\xi| e^{2\nu t|\xi|}|\widehat{\mathcal{D}(\Omega)}(\xi)||\hat{f}(\xi)|d\xi \leq \|f\|_{\dot{H}^{1/2}_\nu}\|\mathcal{D}(\Omega)\|_{\dot{H}^{1/2}_\nu},$$

and using the bounds on $\hat{N}_i$ (4.61) in (4.62) followed by the product rule we obtain that

$$\begin{aligned}
\int e^{2\nu t|\xi|}|\widehat{N_2}(\xi)||\hat{f}(\xi)|d\xi &\leq \sum_{n\geq 1}\int e^{2\nu t|\xi|}|\hat{f}(\xi)|\Big(|\cdot||\hat{\Omega}(\cdot)| * (*^{2n}|\cdot||\hat{f}(\cdot)|)\Big)(\xi)d\xi \\
&\leq \sum_{n\geq 1}\int e^{\nu t|\xi|}|\hat{f}(\xi)|(|\cdot||\hat{\Omega}(\cdot)|e^{\nu t|\cdot|}) * (*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat{f}(\cdot)|)(\xi)d\xi \\
&\leq \sum_{n\geq 1}\int |\xi||\hat{\Omega}(\xi)|e^{\nu t|\xi|}\cdot(e^{\nu t|\cdot|}|\hat{f}(\cdot)|) * (*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat{f}(\cdot)|)d\xi \\
&\leq \sum_{n\geq 1} 2n\int |\xi|^{\frac{1}{2}}|\widehat{\Omega}(\xi)|e^{\nu t|\xi|}\cdot(e^{\nu t|\cdot|}|\hat{f}(\cdot)|) * (|\cdot|^{\frac{3}{2}}e^{\nu t|\cdot|}|\hat{f}(\cdot)|) * (*^{2n-1}|\cdot|e^{\nu t|\cdot|}|\hat{f}(\cdot)|)d\xi \\
&\quad + \sum_{n\geq 1}\int |\xi|^{\frac{1}{2}}|\widehat{\Omega}(\xi)|e^{\nu t|\xi|}\cdot(|\cdot|^{\frac{1}{2}}e^{\nu t|\cdot|}|\hat{f}(\cdot)|) * (*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat{f}(\cdot)|)d\xi \\
&\leq \sum_{n\geq 1} 2n\|\Omega\|_{\dot{H}^{1/2}_\nu}\|f\|_{L^2_\nu}\|f\|_{\dot{\mathcal{F}}^{3/2,1}_\nu}\|f\|^{2n-1}_{\dot{\mathcal{F}}^{1,1}_\nu} + \|\Omega\|_{\dot{H}^{1/2}_\nu}\|f\|_{\dot{H}^{1/2}_\nu}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu} \\
&\leq \sum_{n\geq 1} 2n\frac{\epsilon_n}{2}\|\Omega\|^2_{\dot{H}^{1/2}_\nu} + 2n\frac{1}{2\epsilon_n}\|f\|^2_{L^2_\nu}\|f\|^2_{\dot{\mathcal{F}}^{3/2,1}_\nu}\|f\|^{4n-2}_{\dot{\mathcal{F}}^{1,1}_\nu} \\
&\quad + \|\Omega\|_{\dot{H}^{1/2}_\nu}\|f\|_{\dot{H}^{1/2}_\nu}\|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu},
\end{aligned}$$

where the last line is obtained using Young's inequality for products. We set $\epsilon_n = \epsilon/n^3$ for some small constant $\epsilon > 0$ that we can pick. We can bound the other terms of N_2 and N_3 similarly. It remains to bound $\|\mathcal{D}(\Omega)\|_{\dot{H}^{1/2}_\nu}$ and $\|\Omega\|_{\dot{H}^{1/2}_\nu}$. First,

$$\|\Omega\|_{\dot{H}^{1/2}_\nu} \leq A_\mu \|\mathcal{D}(\Omega)\|_{\dot{H}^{1/2}_\nu} + 2A_\rho \|f\|_{\dot{H}^{1/2}_\nu}.$$

Hence, we need to bound $\|\mathcal{D}(\Omega)\|_{\dot{H}^{1/2}_\nu}$ appropriately:

$$\begin{aligned}\|\mathcal{D}(\Omega)\|_{\dot{H}^{1/2}_\nu} &\leq 2\sum_{n\geq 0} \||\xi|^{\frac{1}{2}} e^{\nu t|\xi|} (*^{2n+1}| \cdot ||\hat{f}(\cdot)|) * |\widehat{\Omega}(\cdot)||_{L^2_\nu} \\ &\leq 2\sum_{n\geq 0} \|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu} \|\Omega\|_{\dot{H}^{1/2}_\nu} + 2(2n+1)\|f\|_{\dot{\mathcal{F}}^{3/2,1}_\nu} \|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu} \|\Omega\|_{L^2_\nu}.\end{aligned}$$

Using this estimate for $\mathcal{D}(\Omega)$,

$$\begin{aligned}\|\Omega\|_{\dot{H}^{1/2}_\nu} \leq (1 - 2A_\mu \sum_{n\geq 0} \|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu})^{-1} \\ \cdot \Big(2A_\mu \sum_{n\geq 0} (2n+1)\|f\|_{\dot{\mathcal{F}}^{3/2,1}_\nu} \|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu} \|\Omega\|_{L^2_\nu} + 2A_\rho \|f\|_{\dot{H}^{1/2}_\nu}\Big).\end{aligned}$$

For $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ of our medium size, the inverted term on the right hand side above is a bounded constant. Also,

$$\begin{aligned}\|\mathcal{D}(\Omega)\|_{\dot{H}^{1/2}_\nu} \leq \sum_{n\geq 0} \|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu} (1 - 2A_\mu \sum_{n\geq 0} \|f\|^{2n+1}_{\dot{\mathcal{F}}^{1,1}_\nu})^{-1} \\ \cdot \Big(2A_\mu \sum_{n\geq 0} (2n+1)\|f\|_{\dot{\mathcal{F}}^{3/2,1}_\nu} \|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu} \|\Omega\|_{L^2_\nu} + 2A_\rho \|f\|_{\dot{H}^{1/2}_\nu}\Big) \\ + (2n+1)\|f\|_{\dot{\mathcal{F}}^{3/2,1}_\nu} \|f\|^{2n}_{\dot{\mathcal{F}}^{1,1}_\nu} \|\Omega\|_{L^2_\nu} \\ \leq C(\|f\|_{\dot{\mathcal{F}}^{1,1}}) \Big(\|f\|_{\dot{\mathcal{F}}^{3/2,1}_\nu} \|\Omega\|_{L^2_\nu} + \|f\|_{\dot{H}^{1/2}_\nu}\Big).\end{aligned}$$

Now, it can be seen that $\|\Omega\|_{L^2_\nu} \leq \tilde{C}(\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|f\|_{L^2_\nu}$ where $\tilde{C}(\|f\|_{\dot{\mathcal{F}}^{1,1}}) \to 0$ as $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu} \to 0$. Thus, summarizing, we can pick $\epsilon > 0$ small enough in the Young's inequality step in the bounds of the integral terms of N_2 and the other nonlinear terms,

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_\nu}(t) \quad \leq \quad \Big(\nu \;-\; A_\rho \;+\; c(\epsilon, \|f\|_{\dot{\mathcal{F}}^{1,1}_\nu})\Big)\|f\|^2_{\dot{H}^{1/2}_\nu} \;+\; \frac{1}{2\epsilon}\tilde{c}(\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu})\|f\|^2_{\dot{\mathcal{F}}^{3/2,1}_\nu}\|f\|^2_{L^2_\nu},$$

where $c(\epsilon, \|f\|_{\dot{\mathcal{F}}^{1,1}}) \to 0$ as $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu} \to 0$ or as $\epsilon \to 0$ and $\tilde{c}(\|f\|_{\dot{\mathcal{F}}^{1,1}}) \to 0$ as $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu} \to 0$. Hence, picking ϵ sufficiently small, but not 0, the first term on the right hand side is negative. By Gronwall's inequality, we obtain

$$\|f\|^2_{L^2_\nu}(t) \leq \|f_0\|^2_{L^2} \exp\Big(\frac{1}{2\epsilon}\tilde{c}(\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t)) \int_0^t \|f\|^2_{\dot{\mathcal{F}}^{3/2,1}_\nu}(\tau)d\tau\Big).$$

Finally, the exponential term on the right hand side is uniformly bounded because by interpolation

$$\int_0^t \|f\|^2_{\dot{\mathcal{F}}^{3/2,1}_\nu}(\tau)d\tau \leq \int_0^t \|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(\tau)\|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)d\tau \leq \|f_0\|_{\dot{\mathcal{F}}^{1,1}} \int_0^t \|f\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)d\tau \leq \|f_0\|^2_{\dot{\mathcal{F}}^{1,1}}.$$

This completes the proof. □

Next, recall the notation

$$\|f\|_{\dot{H}^s_\nu} = \|f\|_{\dot{\mathcal{F}}^{s,2}_\nu} = \int |\xi|^{2s} e^{2|\xi|t\nu} |\hat{f}(\xi)|^2 d\xi.$$

We will use the following inequality on the time derivative of the H^s_ν norm when performing decay estimates in L^2 spaces:

Proposition 4.9.2. *Let* $1/2 \leq s \leq 3/2$ *and assume* $f_0 \in \dot{\mathcal{F}}^{1,1} \cap L^2$ *satisfying* (4.60). *Then,*

$$\frac{d}{dt}\|f\|_{\dot{H}^s_\nu} \leq -C\|f\|_{\dot{H}^{s+1/2}_\nu}. \tag{4.83}$$

Proof. Differentiating the quantity $\|f\|_{\dot{H}^s_\nu}$ and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{\dot{H}^s_\nu} = \nu\|f\|^2_{\dot{H}^{s+1/2}_\nu} - A_\rho\|f\|^2_{\dot{H}^{s+1/2}_\nu} + K_1 + K_2 + K_3,$$

where the terms K_i corresponds to the nonlinear terms N_i in (4.19). Then we have that

$$K_1 \leq \frac{A_\mu}{2}\int |\xi|^{2s} e^{2\nu t|\xi|} |\hat{f}(\xi)||\Lambda\mathcal{D}(\Omega)(\xi)|d\xi.$$

Using the identity $\Lambda = R_1\partial_{\alpha_1} + R_2\partial_{\alpha_2}$, it suffices to prove the following bounds on $\partial_{\alpha_i}\mathcal{D}(\Omega)$:

$$\begin{aligned}\int |\xi|^{2s} e^{2\nu t|\xi|} |\hat{f}(\xi)||R_1\partial_{\alpha_1}\mathcal{D}(\Omega)(\xi)|d\xi &\leq \int |\xi|^{2s} |\hat{f}(\xi)||\partial_{\alpha_1}\mathcal{D}(\Omega)(\xi)|d\xi \\ &\leq \|f\|_{\dot{H}^{s+1/2}_\nu}\|\partial_{\alpha_1}\mathcal{D}(\Omega)\|_{\dot{H}^{s-1/2}_\nu}.\end{aligned}$$

Hence, it suffices to appropriately bound $\|\partial_{\alpha_1}\mathcal{D}(\Omega)\|_{\dot{H}^{s-1/2}_\nu}$. Using (4.23) we have that

$$\begin{aligned}\|\partial_{\alpha_1}\mathcal{D}(\Omega)\|_{\dot{H}^{s-1/2}_\nu} &\leq 2\|BR_1\|_{\dot{H}^{s-1/2}_\nu} + 2\|BR_3\partial_{\alpha_1}f\|_{\dot{H}^{s-1/2}_\nu} \\ &\leq 2(\|BR_1\|_{\dot{H}^{s-1/2}_\nu} + \|BR_3\|_{\dot{H}^{s-1/2}_\nu}\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu} \\ &\qquad + \|f\|_{\dot{H}^{s+1/2}_\nu}\|BR_3\|_{\mathcal{F}^{0,1}_\nu}).\end{aligned}$$

Similarly to previous estimates in Section 4.4, we use the triangle inequality and Young's inequality to obtain that

$$\begin{aligned}\|BR_{11}\|_{\dot H^{s-1/2}_\nu} &\le \frac12\left(\sum_{n\ge0}\left\||\xi|^{s-1/2}e^{\nu t|\xi|}\Big(|\widehat{\omega_3}(\cdot)|*(*^{2n}|\cdot||\hat f(\cdot)|)\Big)(\xi)\right\|_{L^2_\xi}\right)\\
&\le \frac12\left(\sum_{n\ge1}2n\left\|(e^{\nu t|\cdot|}|\widehat{\omega_3}(\cdot)|)*(*^{2n-1}|\cdot|e^{\nu t|\cdot|}|\hat f(\cdot)|)*|\cdot|^{s+1/2}e^{\nu t|\cdot|}|\hat f(\cdot)|\right\|_{L^2}\right)\\
&\quad+\frac12\left(\sum_{n\ge0}\left\|(|\cdot|^{s-1/2}|e^{\nu t|\cdot|}\widehat{\omega_3}(\cdot)|)*(*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat f(\cdot)|)\right\|_{L^2}\right)\\
&\le \frac12\sum_{n\ge1}2n\|\omega_3\|_{\mathcal F^{0,1}_\nu}\|f\|^{2n-1}_{\dot{\mathcal F}^{1,1}_\nu}\|f\|_{\dot H^{s+1/2}_\nu}+\frac12\sum_{n\ge0}\|\omega_3\|_{\dot H^{s-1/2}_\nu}\|f\|^{2n}_{\dot{\mathcal F}^{1,1}_\nu},\end{aligned}$$

and

$$\begin{aligned}\|BR_{12}\|_{\dot H^{s-1/2}_\nu} &\le \frac12\left(\sum_{n\ge0}\left\||\xi|^{s-1/2}e^{\nu t|\xi|}\Big(|\widehat{\omega_2}(\cdot)|*(*^{2n+1}|\cdot||\hat f(\cdot)|)\Big)(\xi)\right\|_{L^2_\xi}\right)\\
&\le \frac12\left(\sum_{n\ge0}(2n+1)\left\|(e^{\nu t|\cdot|}|\widehat{\omega_2}(\cdot)|)*(*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat f(\cdot)|)*|\cdot|^{s+1/2}e^{\nu t|\cdot|}|\hat f(\cdot)|\right\|_{L^2}\right)\\
&\quad+\frac12\left(\sum_{n\ge0}\left\|(|\cdot|^{s-1/2}|e^{\nu t|\cdot|}\widehat{\omega_2}(\cdot)|)*(*^{2n+1}|\cdot|e^{\nu t|\cdot|}|\hat f(\cdot)|)\right\|_{L^2}\right)\\
&\le \frac12\sum_{n\ge0}(2n+1)\|\omega_2\|_{\mathcal F^{0,1}_\nu}\|f\|^{2n}_{\dot{\mathcal F}^{1,1}_\nu}\|f\|_{\dot H^{s+1/2}_\nu}+\frac12\sum_{n\ge0}\|\omega_2\|_{\dot H^{s-1/2}_\nu}\|f\|^{2n+1}_{\dot{\mathcal F}^{1,1}_\nu},\end{aligned}$$

and

$$\begin{aligned}\|BR_3\|_{\dot H^{s-1/2}_\nu} &\le \frac12\left(\sum_{n\ge0}\left\||\xi|^{s-1/2}e^{\nu t|\xi|}\Big((|\widehat{\omega_1}(\cdot)|+|\widehat{\omega_2}(\cdot)|)*(*^{2n}|\cdot||\hat f(\cdot)|)\Big)(\xi)\right\|_{L^2_\xi}\right)\\
&\le \frac12\left(\sum_{n\ge1}2n\left\|(e^{\nu t|\cdot|}|\widehat{\omega_1}(\cdot)|+e^{\nu t|\cdot|}|\widehat{\omega_2}(\cdot)|)*(*^{2n-1}|\cdot|e^{\nu t|\cdot|}|\hat f(\cdot)|)*|\cdot|^{s+1/2}e^{\nu t|\cdot|}|\hat f(\cdot)|\right\|_{L^2}\right.\\
&\quad\left.+\sum_{n\ge0}\left\|(|\cdot|^{s-1/2}|e^{\nu t|\cdot|}\widehat{\omega_1}(\cdot)|+|\cdot|^{s-1/2}|e^{\nu t|\cdot|}\widehat{\omega_2}(\cdot)|)*(*^{2n}|\cdot|e^{\nu t|\cdot|}|\hat f(\cdot)|)\right\|_{L^2}\right)\\
&\le \frac12\sum_{n\ge1}2n(\|\omega_1\|_{\mathcal F^{0,1}_\nu}+\|\omega_2\|_{\mathcal F^{0,1}_\nu})\|f\|^{2n-1}_{\dot{\mathcal F}^{1,1}_\nu}\|f\|_{\dot H^{s+1/2}_\nu}\\
&\quad+\frac12\sum_{n\ge0}(\|\omega_1\|_{\dot H^{s-1/2}_\nu}+\|\omega_2\|_{\dot H^{s-1/2}_\nu})\|f\|^{2n}_{\dot{\mathcal F}^{1,1}_\nu}.\end{aligned}$$

Hence, we now have to prove estimates on $\|\omega_i\|_{\dot{H}^{s-1/2}_\nu}$ for $1/2 \le s \le 3/2$. This follows similar patterns:

$$\|\partial_{\alpha_1}\Omega\|_{\dot{H}^{s-1/2}_\nu} \le 2A_\rho\|f\|_{\dot{H}^{s+1/2}_\nu} + 2A_\mu\|BR_1\|_{\dot{H}^{s-1/2}_\nu} + 2A_\mu\|BR_3\partial_{\alpha_1}f\|_{\dot{H}^{s-1/2}_\nu}.$$

Notice that using the triangle inequality as above on $|\xi|^{s-1/2}$, since $0 \le s - 1/2 \le 1$, we obtain analogously to the steps in Section 4.4 that

$$\|\partial_{\alpha_i}\Omega\|_{\dot{H}^{s-1/2}_\nu} \le 2A_\rho C_{4,\nu}\|f\|_{\dot{H}^{s+1/2}_\nu}.$$

Moreover, for $i = 1, 2$

$$\|\omega_i\|_{\dot{H}^{s-1/2}_\nu} \le 2A_\rho C_{4,\nu}\|f\|_{\dot{H}^{s+1/2}_\nu},$$

and

$$\|\omega_3\|_{\dot{H}^{s-1/2}_\nu} \le 4A_\mu A_\rho\|f\|^2_{\mathcal{F}^{1,1}}(C_{5,\nu} + 3C_{2,\nu})\|f\|_{\dot{H}^{s+1/2}_\nu}.$$

Now we can follow the steps in Proposition 4.5.1. Plugging in the estimates above and performing similar estimates for K_2 and K_3, we obtain for $1/2 \le s \le 3/2$

$$\frac{d}{dt}\|f\|_{\dot{H}^s_\nu} \le -C\|f\|_{\dot{H}^{s+1/2}_\nu}, \tag{4.84}$$

for a positive constant C depending on f_0 and ν. □

4.10 Large-Time Decay of Analytic Norms

In this section, we begin by proving the Decay Lemma we will use to show large time decay of solutions to the Muskat problem:

Lemma 4.10.1 (Decay Lemma). *Suppose* $\|g\|^p_{\dot{\mathcal{F}}^{s_1,p}_\nu}(t) \le C_0$ *and*

$$\frac{d}{dt}\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu}(t) \le -C\|g\|^p_{\dot{\mathcal{F}}^{s_2+1/p,p}_\nu}(t) \tag{4.85}$$

such that $s_1 \le s_2$ *and* $p \in [1, \infty)$. *Then*

$$\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu}(t) \lesssim (1+t)^{(s_1-s_2)p}.$$

Proof. Consider $r > 0$. Then

$$\begin{aligned}
\|g\|^p_{\dot{\mathcal{F}}^{r,p}_\nu} &= \int e^{\nu tp|\xi|}|\xi|^{rp}|\hat{g}(\xi)|^p d\xi \\
&\ge \int_{|\xi|>(1+\delta t)^s} e^{\nu tp|\xi|}|\xi|^{rp}|\hat{g}(\xi)|^p d\xi \\
&\ge (1+\delta t)^s \int_{|\xi|>(1+\delta t)^s} e^{\nu tp|\xi|}|\xi|^{(r-1/p)p}|\hat{g}(\xi)|^p d\xi \\
&= (1+\delta t)^s \Big(\|g\|^p_{\dot{\mathcal{F}}^{r-1/p,p}_\nu} - \int_{|\xi|\le(1+\delta t)^s} e^{\nu tp|\xi|}|\xi|^{(r-1/p)p}|\hat{g}(\xi)|^p d\xi\Big).
\end{aligned}$$

We can use (4.85) and the above argument with $r = s_2 + 1/p$ to obtain that

$$\begin{aligned}\frac{d}{dt}\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu} + C(1+\delta t)^s\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu} &\le -C\|g\|^p_{\dot{\mathcal{F}}^{s_2+1/p,p}_\nu} + C(1+\delta t)^s\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu}\\ &\le C(1+\delta t)^s\Big(\int_{|\xi|\le(1+\delta t)^s} e^{\nu tp|\xi|}|\xi|^{s_2p}|\hat{g}(\xi)|^p d\xi\Big)\\ &\le C(1+\delta t)^{s(s_2-s_1)p}(1+\delta t)^s\Big(\int_{|\xi|\le(1+\delta t)^s} e^{\nu tp|\xi|}|\xi|^{s_1p}|\hat{g}(\xi)|^p d\xi\Big)\\ &\le C(1+\delta t)^{s(s_2-s_1)p}(1+\delta t)^s\|g\|^p_{\dot{\mathcal{F}}^{s_1,p}_\nu}\\ &\le CC_0(1+\delta t)^{s(s_2-s_1)p}(1+\delta t)^s.\end{aligned}$$

Now, let $\sigma > (s_2 - s_1)p$ and choose δ such that $\delta\sigma = C$, $s = -1$. Then

$$\begin{aligned}\frac{d}{dt}((1+\delta t)^\sigma\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu}) &= (1+\delta t)^\sigma\frac{d}{dt}\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu} + \sigma\delta\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu}(1+\delta t)^{\sigma-1}\\ &= (1+\delta t)^\sigma\frac{d}{dt}\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu} + C\|g\|_{\dot{\mathcal{F}}^{s_2,p}_\nu}(1+\delta t)^{\sigma-1}\\ &= (1+\delta t)^\sigma(\frac{d}{dt}\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu} + C\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu}(1+\delta t)^{-1})\\ &\le CC_0(1+\delta t)^{\sigma-(s_2-s_1)p-1}.\end{aligned}$$

Integrating in time we obtain that

$$(1+\delta t)^\sigma\|g\|^p_{\dot{\mathcal{F}}^{s_2,p}_\nu} \le \frac{\tilde{C}}{\delta t}(1+(1+\delta)^{\sigma-(s_2-s_1)p})$$

for some constant $\tilde{C}$. Dividing both sides of the inequality by $(1+\delta t)^\sigma$ we obtain our results. □

We can now use this lemma to prove large-time decay rates for the analytic norms. By Holder's inequality, for $s > -d/2$ and $r > s + d/2$

$$\begin{aligned}\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu} &\le \|f\|_{H^r_\nu}\Big\|\frac{|\xi|^s}{(1+|\xi|^2)^{r/2}}\Big\|_{L^1}\\ &\lesssim \|f\|_{H^r_\nu}\end{aligned}$$

Hence, by the estimate (4.83), we obtain for $-1 < s' < 0$:

$$\|f\|_{\dot{\mathcal{F}}^{s',1}_\nu}(t) \le C_s \tag{4.86}$$

for a fixed constant C_s. By the Decay Lemma, this implies that

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu} \lesssim (1+t)^{-1-s+\lambda} \tag{4.87}$$

for $0 \le s \le 1$ and arbitrarily small $\lambda > 0$. This proves Theorem 4.3.4. We can further demonstrate decay of other analytic norms. For example, the quantities $\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}$ for $s > 1$ also decay in time. First, note that

$$\|f\|_{\dot{\mathcal{F}}^{s,1}}(t) \le \|e^{-\nu t|\xi|}|\xi|^{s-1}\|_{L^\infty_\xi}\|f\|_{\mathcal{F}^{0,1}_\nu} \le \|e^{-\nu t|\xi|}|\xi|^{s-1}\|_{L^\infty_\xi}k(A_\mu) < \infty$$

for any $t > 0$. Moreover, using the weighted triangle inequality

$$|\xi|^s \leq (n+1)^s(|\xi - \xi_1|^s + |\xi_1 - \xi_2|^s + \cdots + |\xi_n|^s),$$

it can be seen that

$$\frac{d}{dt}\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}(t) \leq -C(s)\|f\|_{\dot{\mathcal{F}}^{s+1,1}_\nu}(t)$$

for a positive constant $C(s)$ depending on s and $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ when $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}$ is sufficiently small. However, by (4.87) for $s = 1$, the quantity $\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t)$ does indeed decay to a sufficiently small quantity when $t > T_s$ for some $T_s > 0$ depending on s and the initial data f_0. Hence, since, as noted earlier, $\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu}(T_s) < \infty$, we can apply the Decay Lemma to obtain

Theorem 4.10.2. *Let $s \geq 1$. Then, $\|f\|_{\dot{\mathcal{F}}^{s,1}}(t) < \infty$ for all $t > 0$ and we have the decay*

$$\|f\|_{\dot{\mathcal{F}}^{s,1}_\nu} \leq C_{1,s}t^{-s-1}$$

for $t > T_s$ and $C_{1,s}$ depending on s and the initial data f_0.

We can similarly see by the uniform bounds of Theorem 4.9.1, that $\|f\|_{\dot{H}^s}(t) < \infty$ for all $s \geq 0$ and $t > 0$. Hence, by similar arguments to above and Proposition 4.9.2, the Decay Lemma 4.10.1 with $p = 2$,

Theorem 4.10.3. *Suppose $s \geq 1/2$. Then $\|f\|_{\dot{H}^s} < \infty$ for all $t > 0$ and we have the decay*

$$\|f\|^2_{\dot{H}^s_\nu} \leq C_{2,s}t^{-2s}$$

for $t > T_s$ and $C_{2,s}$ depending on s and the initial data f_0.

4.11 Ill-posedness

We will make use of the results previously shown to prove that the Muskat problem in the unstable case $\rho_1 > \rho_2$ is ill-posed for any Sobolev space H^s with $s > 0$. First we notice that our initial data $f \in L^2 \cap \dot{\mathcal{F}}^{1,1}$ need not be in H^s.

Lemma 4.11.1. *There exists a function $f \in L^2 \cap \dot{\mathcal{F}}^{1,1}$ with*

$$\|f_0\|_{L^2} < \infty, \quad \|f_0\|_{\dot{\mathcal{F}}^{1,1}} < k_\mu$$

for a constant k_μ of medium size such that $f \notin H^s$ for any $s > 0$.

Proof. We give an explicit counterexample. Consider a radial function $f : \mathbb{R}^2 \to \mathbb{R}$. Let for $n \geq N$ for some $N > 0$ integer

$$|\xi\hat{f}(\xi)| = r|\hat{f}(r)| = \begin{cases} n^\sigma & \text{if } r \in [n^\delta, n^\delta + 1/n^\gamma] \\ 0 & \text{otherwise,} \end{cases}$$

where σ, δ and γ are positive. Then one can compute

$$\|f\|_{\dot{\mathcal{F}}^{1,1}} = 2\pi \int_0^\infty r^2 |\hat{f}(r)| dr \leq 2\pi \sum_{n\geq N} n^{\sigma+\delta-\gamma} + n^{\sigma-2\gamma} < \infty$$

when $\sigma + \delta - \gamma < -1$. For $0 \leq s < 1/2$ we have that

$$\|f\|_{\dot{H}^s}^2 = \int_0^\infty (r|\hat{f}(r)|)^2 r^{2s-1} dr$$

and having chosen $N > 0$ appropriately large,

$$\begin{aligned} 2^{2s-1} 2\pi \sum_{n\geq N} n^{2\sigma+\delta(2s-1)-\gamma} &\leq 2\pi \sum_{n\geq N} n^{2\sigma-\gamma}(n^\delta + n^{-\gamma})^{2s-1} \\ &\leq 2\pi \int_0^\infty (r|\hat{f}(r)|)^2 r^{2s-1} dr \\ &\leq 2\pi \sum_{n\geq N} n^{2\sigma+\delta(2s-1)-\gamma}. \end{aligned}$$

Hence, pick σ, δ and γ such that $2\sigma + \delta(2s-1) - \gamma = -1$. Then, $\|f\|_{L^2} < +\infty$ and for $s > 0$, $\|f\|_{\dot{H}^s} = +\infty$. This counterexample gives the proof in the 3D case, as we can force $\|f\|_{\dot{\mathcal{F}}^{1,1}} < k_\mu$ by multiplying this counterexample by the appropriate constant.

In the 1D interface case, let for $n \geq N$ for some $N > 0$ integer

$$\xi \hat{f}(\xi) = \begin{cases} n^\sigma & \text{if } \xi \in [n^\delta, n^\delta + 1/n^\gamma] \\ 0 & \text{otherwise} \end{cases}$$

such that $\gamma > \sigma + 1$, $2\delta + \gamma > 2\sigma + 1$ but $2\delta(1-s) + \gamma = 2\sigma + 1$. Then one can compute that

$$\|f\|_{L^2} < \infty, \|f\|_{\dot{\mathcal{F}}^{1,1}} < k_\mu \text{ and } \|f\|_{H^s} = +\infty.$$

□

Remark 4.11.2. *This example can be adapted to show that even if $f \in \dot{\mathcal{F}}_\nu^{1,1} \cap L^2$, it need not be in H^s.*

Theorem 4.11.3. *For every $\epsilon > 0$, there exists a solution $\tilde{f}$ to the unstable regime and $0 < \delta < \epsilon$ such that $\|\tilde{f}\|_{H^s}(0) = \delta$ but $\|\tilde{f}\|_{H^s}(\delta) = \infty$.*

Proof. Take $f_0 \in L^2 \cap \dot{\mathcal{F}}^{1,1}$ satisfying condition (4.60) for the Muskat problem in the stable regime such that $\|f_0\|_{H^s} = \infty$. By the gain of regularity in (4.35)

$$\|f\|_{H^s}(\delta) \leq \|e^{-\nu\delta|\xi|}|\xi|^s\|_{L^\infty} \|f\|_{L^2_\nu}(\delta) \leq c(\delta)\|f_0\|_{L^2} \exp\left(R(\|f_0\|_{\dot{\mathcal{F}}^{1,1}})\right) < \epsilon$$

by picking initial data with $\|f_0\|_{L^2}$ sufficiently small. If $f(x,t)$ is a solution to the stable case problem, then $\tilde{f}(x,t) = f(x, -t+\delta)$ is a solution to the unstable case $\rho_1 > \rho_2$. Hence,

$$\|\tilde{f}\|_{H^s}(0) = \|f\|_{H^s}(\delta) < \epsilon \text{ and } \|\tilde{f}\|_{H^s}(\delta) = \|f\|_{H^s}(0) = \infty.$$

□

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