# Research Statement

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My research interests lie within analysis, partial differential equations and fluid mechanics. More specifically, I am interested in the evolution of incompressible fluid interfaces coming from physical models. During my graduate career, I have studied the global well-posedness and interface regularity for patch problems in nonlinear parabolic equations such as Navier-Stokes [6] and Boussinesq [5]. I have also worked on the Muskat problem [7], addressing additional questions like large-time decay of solutions or ill-posedness.

## 1. Overview

In the following I proceed to expose an overview of my main research topics, contributions and plans for future investigation.

### 1.1. Inhomogeneous Navier-Stokes.

The inhomogeneous Navier-Stokes equations model physical systems in which fluids of non-constant densities move as an incompressible flow. This regime is the case in many geophysical problems, such as stratified fluids in oceans. More prominently, these equations are used to describe the dynamics of a system of two or more immiscible liquids. Mathematically, the theory of strong solutions for *INS* is still not complete even in the 2D case, while the 3D case encompasses the well-known Navier-Stokes Millennium problem.

Indeed, results allowing discontinuous density in 2D have only appeared recently. Nevertheless, the so-called *density patch* problem, proposed by P.-L. Lions in 1996 [8], remained as a particularly challenging open problem. The principle challenge of this problem is to establish if the smoothness of the initial interface given by the patch is preserved in time. In [6] we give a positive answer when the initial boundary between the two liquids has $C^{1+\gamma}$ regularity or greater, for any density jump and any size of the initial velocity. More remarkably, our proof also works in the limit case of $W^{2,\infty}$, thus providing control of the curvature.

To propagate regularity through the mass conservation equation, a gain of regularity for the velocity is first needed. As the density is given by a patch, and hence is of low regularity, the quasilinear coupling between the density and the velocity makes the parabolic gain of regularity difficult to achieve by standard procedures. Thus, the use of time-weighted energy estimates and the higher regularity of the convective derivative are crucial in this step. By a combination of different techniques, including the characterization as a multiplier in Besov spaces of characteristic functions, the extra cancellations of singular integrals with even kernels, commutators estimates and level-set methods, we can bootstrap our result from low regular initial interfaces to higher ones, thus providing a unified approach.

For further research, I deem the strategy introduced in [6] as a promising first step to tackle the more physical situation where viscosity jump is included. In the one-fluid case, i.e., water waves, splash singularities are known to appear in finite time even for viscous fluids [1],[2]. Moreover, the presence of a second fluid precludes this kind of singularity [4], although very little is known with respect to global regularity in this two-fluid interface case. I intend to show that the evolution of a two viscous fluid interface preserves the initial regularity globally in time. This would greatly

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advance our understanding of the causes of interface collapse and loss of regularity. As a long-term problem, the analogous question can be posed for an interface between viscous and inviscid fluids. Further work on aforementioned techniques of [6] will produce new results in the case in which one of the densities is zero as well as for models with fractional dissipation. There are many other problems for possible improvements in research, in particular, substituting one of the fluids by an elastic solid, which would lead to an hiperbolic-parabolic problem.

### 1.2. Boussinesq system

In natural convection phenomena, i.e., when the fluid motion is induced by temperature gradients without external sources, density variations are usually negligible in inertia terms. This leads to the Boussinesq approximation, widely used in meteorology and in building environment models. From the mathematical point of view, the main interest lies on its connection to the 3D Navier-Stokes and Euler equations since it captures the phenomenon of vortex stretching. Indeed, in the inviscid and non-diffusive setting, the well-posedness of the 2D Boussinesq system remains as an outstanding open problem.

When diffusion is neglected, the energy equation reduces to a transport one. Therefore, initial temperature patches are preserved in time. Moreover, as we prove in [5], the curvature of the boundary of a patch cannot blow up in finite time. We also give analogous global-in-time results for lower and higher regular interfaces measured in Hölder spaces. For the lower regularity regime, we make use of the parabolic structure of the model by applying maximal regularity results for the heat equation. Building upon this result, a new cancellation found in the singular integral parabolic operators given by two spatial derivatives of the heat kernels and its combination with Riesz transforms allows us to gain control of the curvature. Going one step further, we take advantage of the extra regularity of the velocity in the tangential direction to propagate more regular interfaces.

For future work, I would like to extend these results to 3D. As this model contains 3D Navier-Stokes as a particular case, I propose to study global regularity for small initial data and local regularity for arbitrary initial data. This second result would lead to a scenario where looking for singularities is appropriate. These singularities would provide a rich understanding of the fundamental difference between the 2D and 3D settings, as splash singularities have not been found in the two-fluid case for 3D Navier-Stokes.

### 1.3. Porous medium equation

The motion of two incompressible immiscible fluids through a porous medium gives rise to a notable free boundary problem called Muskat problem. Muskat first derived it based on the experimental Darcy's law to model the behavior of water through oil sand in typical pumping processes in petroleum industry. In [7] we study global-in-time existence, uniqueness and regularity of solutions in critical spaces in the stable regime, i.e., when the more dense fluid is at the bottom. We do it for the actual physical case where the fluids have different densities and viscosities, and for initial slopes not necessarily small but just bounded by an explicit constant. Moreover, we improved the previous known constants for the non-viscosity jump case [3]. In addition, we show that the interface becomes instantly analytic and flattens in time, giving large-time decay rates for the solution. We also show ill-posedness in unstable situations even for low regular solutions.

In the aforementioned result zero surface tension is assumed. By incorporating capillary forces, the problem becomes locally well-posed independently of the relatively positions of the fluids. Although surface tension acts as a restoring force for the interface, its highly nonlinear nature gives rise to interesting new physical phenomena such as viscous fingering. I would like to study deeper this new regime with focus on global well-posedness and large-time decay rates.

## References


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